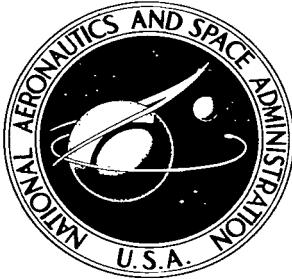


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NUMERICAL SOLUTION OF AN INTEGRAL EQUATION ON THE THEORY OF LIGHT SCATTERING IN THE ATMOSPHERE

by Ye. S. Kuznetsov and B. V. Ovchinskij

Trudy Geofizicheskogo Instituta, No. 4, 1949.

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ON THE THEORY OF LIGHT SCATTERING IN THE ATMOSPHERE

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NUMERICAL SOLUTION OF AN INTEGRAL EQUATION ON THE THEORY
OF LIGHT SCATTERING IN THE ATMOSPHERE

*2

Ye.S.Kuznetsov and B.V.Ovchinskiy

Individual papers by various authors discuss the results of computation of spherical scattering of light in the atmosphere by integral equations, with emphasis on tabulated data. Eight- and seven-place Tables are given for the $E_1(X)$ to $E_4(X)$ functions, based on various values of physical parameters such as optical thickness of the atmosphere, zenith distance of the sun, and albedo. Successive approximations, including the zero-th approximation, are derived and corrections for errors of quadrature with allowance for haze, diffusion, and reflection are given.

PREFACE

This study is part of a large computation project on the application of the theory of atmospheric light scattering undertaken by the Mathematical Department, Geophysical Institute, USSR Academy of Sciences, based on the theory developed by the Department in 1943-44**.

The entire computational material is divided into two parts, each separately developed. This is the Part I, consisting principally of the results of a 1/3 numerical solution of the integral equation of the theory of light scattering in the atmosphere for various values of the physical parameters, including also auxiliary Tables for various purposes, together with Tables of the haze factor. All Tables in this Part are of independent interest, and may be used as the basis for calculations of various practical and theoretical problems connected with the scattering of light in the atmosphere.

The Tables are accompanied by text - six individual papers by various authors, with the primary object of giving as complete an idea as possible of the numerical methods used in compiling the Tables. Most of the computational methods and results given in these papers are original and were developed specifically for the computational work involved.

Part II will include Tables relating to various concrete applications of the theory of atmospheric light scattering.

* Numbers in the margin indicate pagination in the original foreign text.

** Cf. papers by Ye.S.Kuznetsov in Izv. AN SSSR, ser. geog. i geofiz. Vol.7, pp.247-336, 1943; Vol.9, pp.63-72, and pp.204-223, 1945.

All results of calculations in both Part I and Part II apply to the case of spherical scattering. Nonspherical scattering will be the subject of a separate computing project, and the results will be separately reported.

The numerical methods used in making up the Tables were developed by Ye.S. Kuznetsov (Papers I, II, IV, VI) and B.V.Ovchinskiy (Papers III and V). Ye.S. Kuznetsov was in general charge of the work. The computational work was performed by Yu.A.Gulin and Ye.V.Kuznetsova, mathematicians of the Mathematical Department of the Institute, under the direct supervision by B.V.Ovchinskiy.

We wish to thank Prof.I.A.Khvostikov for his assistance in organizing the work and for his constant support throughout the investigation, which contributed to its successful conclusion.

Moscow
May, 1948

Ye.S.Kuznetsov
B.V.Ovchinskiy

I. TABLES OF THE FUNCTION $E_n(x)$

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Ye.S.Kuznetsov

The solution of the integral equation of the light-scattering theory is intimately connected with the theory of the special class of transcendental functions defined by the integral

$$E_n(x) = \int_1^{\infty} \frac{e^{-sx}}{s^n} ds \quad (x \geq 0; n \geq 0). \quad (1)$$

For $n = 0$, this yields the function

$$E_0(x) = \frac{e^{-x}}{x}. \quad (2)$$

For $n = 1$, we arrive at the function

$$E_1(x) = \int_1^{\infty} \frac{e^{-sx}}{s} ds, \quad (3)$$

which is connected with the well-known function

$$Ei(x) = \int_{-\infty}^x \frac{e^s ds}{s}. \quad (4)$$

by the relation

$$E_1(x) = -Ei(-x). \quad (5)$$

Solution of the integral equation of light scattering by the method developed by us requires the use of Tables of the functions $E_1(x)$, $E_2(x)$, $E_3(x)$ and, for certain special calculations, also of the functions $E_4(x)$. For non-spherical scattering, Tables of functions $E_n(x)$ and higher orders become necessary. Available published Tables were found inadequate both in completeness and in number of significant figures. We therefore initiated a special project for the computation of Tables of functions $E_n(x)$. We now present Tables of the functions $E_1(x)$, $E_2(x)$, $E_3(x)$ and $E_4(x)$ for values of the argument for each hundredth from 0.00 to 0.60. The Tables are eight-place for the function $E_1(x)$ and seven-place for the others. The accuracy and detail of the Tables were based on the requirements of the method adopted by us for the numerical solution of the integral equation of the theory of light scattering, and the scope 15 of the Tables (from 0.00 to 0.60) was dictated by the values of the physical parameters used (optical thickness of the atmosphere) and satisfactorily meets the requirements of atmospheric optics.

Below, we present explanations of the technique of calculating these Tables. First, we give a summary of the formulas relating to the theory of the functions $E_n(x)$ which are used in the mathematical theory of radiation, in compiling Tables of these functions, and in various types of numerical solutions of problems connected with radiation theory. Some of the formulas here were taken from published sources and others were independently derived by us.

1. Properties of the Function $E_1(x)$

$$E_1(x) = O\left(\log \frac{1}{x}\right), \quad x \rightarrow 0; \quad (6)$$

$$E_1(x) = O\left(\frac{e^{-x}}{x}\right), \quad x \rightarrow \infty; \quad (7)$$

$$E_1(x) = -c + \log \frac{1}{x} + x - \frac{1}{2} \frac{x^2}{2!} + \frac{1}{3} \frac{x^3}{3!} - \frac{1}{4} \frac{x^4}{4!} + \dots, \quad (8)$$

$$c = 0.5772156649015\dots$$

For greater values of x , we have the asymptotic expansion

$$E_1(x) \sim e^{-x} \left(\frac{1}{x} - \frac{1}{x^2} + \frac{1 \cdot 2}{x^3} - \frac{1 \cdot 2 \cdot 3}{x^4} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{x^5} - \dots \right). \quad (9)$$

2. Properties of the Function $E_n(x)$

$$E_n(0) = \frac{1}{n-1}, \quad (n > 1); \quad (10)$$

$$E_n(x) = O\left(\frac{e^{-x}}{x}\right), \quad x \rightarrow \infty; \quad (11)$$

$$E_n(x) \sim e^{-x} \left(\frac{1}{x} - \frac{n}{x^2} + \frac{n(n+1)}{x^3} - \frac{n(n+1)(n+2)}{x^4} + \dots \right). \quad (12)$$

3. Inequalities

$$E_{n+1}(x) < E_n(x); \quad (13)$$

$$(n-1)E_n(x) < nE_{n+1}(x), \quad (n > 1); \quad (14)$$

$$\frac{e^{-x}}{x+n} < E_n(x) < \frac{e^{-x}}{x+n-1}, \quad (n \geq 1); \quad (15)$$

$$E_n^2(x) < E_{n-1}(x)E_{n+1}(x), \quad (16)$$

$$\frac{d}{dx} \left\{ \frac{E_{n+1}(x)}{E_n(x)} \right\} > 0; \quad (17)$$

$$\frac{d}{dx} \left\{ \frac{E_n(x-a)}{E_n(x)} \right\} < 0, \quad (x > a > 0). \quad (18)$$

4. Relations between Functions $E_n(x)$ of Different Orders

16.

$$n E_{n+1}(x) = e^{-x} - x E_n(x); \quad (19)$$

$$E_{n+1}(x) = \int_x^{\infty} E_n(x) dx; \quad (20)$$

$$\frac{dE_{n+1}(x)}{dx} = -E_n(x); \quad (21)$$

$$E_2(x) = e^{-x} - x E_1(x); \quad (22)$$

$$E_3(x) = \frac{1}{2} [e^{-x} (1-x) + x^2 E_1(x)]; \quad (23)$$

$$E_4(x) = \frac{1}{6} [e^{-x} (2-x+x^2) - x^3 E_1(x)]; \quad (24)$$

$$E_5(x) = \frac{1}{24} [e^{-x} (6-2x+x^2-x^3) + x^4 E_1(x)]; \quad (25)$$

$$E_6(x) = \frac{1}{120} [e^{-x} (24-6x+3x^2-x^3+x^4) - x^5 E_1(x)]; \quad (26)$$

.....

$$E_n(x) = \frac{1}{(n-1)!} \{ e^{-x} [(n-2)! - (n-3)!x + (n-4)!x^2 - \dots \\ \dots + (-1)^{n-2}x^{n-2}] + (-1)^{n-1}x^{n-1} E_1(x) \}. \quad (27)$$

5. Integrals Containing the Functions $E_n(x)$

$$\int_0^x t^m E_n(t) dt = m! \left\{ \frac{1}{m+n} - \left[\frac{x^m}{m!} E_{n+1}(x) + \frac{x^{m-1}}{(m-1)!} E_{n+2}(x) + \dots \right. \right. \\ \left. \left. \dots + \frac{x}{1!} E_{n+m}(x) + E_{n+m+1}(x) \right] \right\}; \quad (28)$$

$$\int_0^x E_n(t) dt = \frac{1}{n} - E_{n+1}(x); \quad (29)$$

$$\int_0^x tE_n(t) dt = \frac{1}{n+1} - [xE_{n+1}(x) + E_{n+2}(x)]; \quad (30)$$

$$\int_0^x t^2 E_n(t) dt = 2 \left\{ \frac{1}{n+2} - \left[\frac{x^2}{2} E_{n+1}(x) + xE_{n+2}(x) + E_{n+3}(x) \right] \right\}; \quad (31)$$

$$\begin{aligned} \int_0^x t^3 E_n(t) dt &= 6 \left\{ \frac{1}{n+3} - \left[\frac{x^3}{6} E_{n+1}(x) + \right. \right. \\ &\quad \left. \left. + \frac{x^2}{2} E_{n+2}(x) + xE_{n+3}(x) + E_{n+4}(x) \right] \right\}; \end{aligned} \quad (32)$$

$$\begin{aligned} \int_0^x t^4 E_n(t) dt &= 24 \left\{ \frac{1}{n+4} - \left[\frac{x^4}{24} E_{n+1}(x) + \frac{x^3}{6} E_{n+2}(x) + \right. \right. \\ &\quad \left. \left. + \frac{x^2}{2} E_{n+3}(x) + xE_{n+4}(x) + E_{n+5}(x) \right] \right\}; \end{aligned} \quad (33)$$

.....

$$\int_0^x E_1(t) dt = 1 - E_2(x); \quad (34)$$

$$\int_0^x tE_1(t) dt = \frac{1}{2} - [xE_2(x) + E_3(x)]; \quad (35)$$

$$\int_0^x t^2 E_1(t) dt = 2 \left\{ \frac{1}{3} - \left[\frac{x^2}{2} E_2(x) + xE_3(x) + E_4(x) \right] \right\}; \quad (36)$$

$$\int_0^x t^3 E_1(t) dt = 6 \left\{ \frac{1}{4} - \left[\frac{x^3}{6} E_2(x) + \frac{x^2}{2} E_3(x) + xE_4(x) + E_5(x) \right] \right\}; \quad (37)$$

$$\begin{aligned} \int_0^x t^4 E_1(t) dt &= 24 \left\{ \frac{1}{5} - \left[\frac{x^4}{24} E_2(x) + \frac{x^3}{6} E_3(x) + \frac{x^2}{2} E_4(x) + \right. \right. \\ &\quad \left. \left. + xE_5(x) + E_6(x) \right] \right\}; \end{aligned} \quad (38)$$

.....

$$\begin{aligned} \int_0^x t^m E_1(t) dt &= m! \left\{ \frac{1}{m+1} - \left[\frac{x^m}{m!} E_2(x) + \frac{x^{m-1}}{(m-1)!} E_3(x) + \dots \right. \right. \\ &\quad \left. \left. + xE_{m+1}(x) + E_{m+2}(x) \right] \right\}; \end{aligned} \quad (39)$$

$$\begin{aligned} \int_0^x t^m E_2(t) dt &= m! \left\{ \frac{1}{m+2} - \left[\frac{x^m}{m!} E_3(x) + \frac{x^{m-1}}{(m-1)!} E_4(x) + \dots \right. \right. \\ &\quad \left. \left. + xE_{m+2}(x) + E_{m+3}(x) \right] \right\}; \end{aligned} \quad (40)$$

$$\begin{aligned} \int_0^x t^m E_3(t) dt &= m! \left\{ \frac{1}{m+3} - \left[\frac{x^m}{m!} E_4(x) + \frac{x^{m-1}}{(m-1)!} E_5(x) + \dots \right. \right. \\ &\quad \left. \left. \dots + xE_{m+3}(x) + E_{m+4}(x) \right] \right\}; \end{aligned} \quad (41)$$

$$\begin{aligned} \int_0^x t^m E_4(t) dt &= m! \left\{ \frac{1}{m+4} - \left[\frac{x^m}{m!} E_5(x) + \frac{x^{m-1}}{(m-1)!} E_6(x) + \dots \right. \right. \\ &\quad \left. \left. \dots + xE_{m+4}(x) + E_{m+5}(x) \right] \right\}; \end{aligned} \quad (42)$$

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$$\int_0^a E_1(|x-t|) dt = 2 - E_2(x) - E_2(a-x); \quad (43)$$

$$\begin{aligned} \int_0^a t E_1(|x-t|) dt &= x [2 - E_2(x) - E_2(a-x)] + xE_2(x) + E_3(x) - \\ &\quad - [(a-x)E_2(a-x) + E_3(a-x)] = \\ &= 2x + E_3(x) - E_3(a-x) - aE_2(a-x); \end{aligned} \quad (44)$$

$$\begin{aligned} \int_0^a t^2 E_1(|x-t|) dt &= x^2 [2 - E_2(x) - E_2(a-x)] + \\ &\quad + 2x \{xE_2(x) + E_3(x) - [(a-x)E_2(a-x) + E_3(a-x)]\} + \\ &\quad + 2 \left\{ \frac{2}{3} - \left[\frac{x^2}{2} E_2(x) + xE_3(x) + E_4(x) \right] - \right. \\ &\quad \left. - \left[\frac{(a-x)^2}{2} E_2(a-x) + (a-x)E_3(a-x) + E_4(a-x) \right] \right\}; \end{aligned} \quad (45)$$

$$\begin{aligned} \int_0^a t^3 E_1(|x-t|) dt &= x^3 [2 - E_2(x) - E_2(a-x)] + 3x^2 \{xE_2(x) + \\ &\quad + E_3(x) - [(a-x)E_2(a-x) + E_3(a-x)]\} + \\ &\quad + 6x \left\{ \frac{2}{3} - \left[\frac{x^2}{2} E_2(x) + xE_3(x) + E_4(x) \right] - \right. \\ &\quad \left. - \left[\frac{(a-x)^2}{2} E_2(a-x) + (a-x)E_3(a-x) + E_4(a-x) \right] \right\} + \\ &\quad + 6 \left\{ \frac{x^3}{6} E_2(x) + \frac{x^2}{2} E_3(x) + xE_4(x) + E_5(x) - \right. \\ &\quad \left. - \left[\frac{(a-x)^3}{6} E_2(a-x) + \frac{(a-x)^2}{2} E_3(a-x) + (a-x)E_4(a-x) + E_5(a-x) \right] \right\}; \end{aligned} \quad (46)$$

$$\begin{aligned}
\int_0^a t^4 E_1(|x-t|) dt &= x^4 [2 - E_2(x) - E_2(a-x)] + 4x^3 \{xE_3(x) + E_3(x) + \\
&+ [(a-x)E_2(a-x) + E_3(a-x)]\} + 12x^2 \left\{ \frac{2}{3} - \left[\frac{x^2}{2} E_2(x) + xE_3(x) + E_4(x) \right] - \right. \\
&- \left[\frac{(a-x)^2}{2} E_2(a-x) + (a-x)E_3(a-x) + E_4(a-x) \right] \} + \\
&+ 24x \left\{ \frac{x^3}{6} E_2(x) + \frac{x^2}{2} E_3(x) + xE_4(x) + E_5(x) - \right. \\
&- \left[\frac{(a-x)^3}{6} E_2(a-x) + \frac{(a-x)^2}{2} E_3(a-x) + \right. \\
&\left. \left. - x\right] + E_5(a-x) \right\} + \\
&+ 24 \left\{ \frac{2}{5} - \left[\frac{x^4}{24} E_2(x) + \frac{x^3}{6} E_3(x) + \frac{x^2}{2} E_4(x) + xE_5(x) + E_6(x) \right] - \right. \\
&- \left[\frac{(a-x)^4}{24} E_2(a-x) + \frac{(a-x)^3}{6} E_3(a-x) + \frac{(a-x)^2}{2} E_4(a-x) + \right. \\
&\left. \left. + (a-x)E_5(a-x) + E_6(a-x) \right] \right\}; \quad (47)
\end{aligned}$$

.....

$$\begin{aligned}
\int_0^a t^m E_1(|x-t|) dt &= x^m \{2 - E_2(x) - E_2(a-x)\} + mx^{m-1} \{xE_2(x) + \\
&+ E_3(x) - [(a-x)E_2(a-x) + E_3(a-x)]\} + \\
&+ m(m-1)x^{m-2} \left\{ \frac{2}{3} - \left[\frac{x^2}{2} E_2(x) + xE_3(x) + E_4(x) \right] - \right. \\
&- \left[\frac{(a-x)^2}{2} E_2(a-x) + (a-x)E_3(a-x) + E_4(a-x) \right] \} + \\
&+ m(m-1)\dots 2 \cdot 1 \left\{ (-1)^m \left[\frac{1}{m+1} - \left(\frac{x^m}{m!} E_2(x) + \dots + xE_{m+1}(x) + \right. \right. \right. \\
&\left. \left. \left. + E_{m+2}(x) \right) \right] + \left[\frac{1}{m+1} - \left(\frac{(a-x)^m}{m!} E_2(a-x) + \dots + \right. \right. \right. \\
&\left. \left. \left. + (a-x)E_{m+1}(a-x) + E_{m+2}(a-x) \right) \right] \right\}; \quad (48)
\end{aligned}$$

$$\int_0^x e^{-kt} E_1(t) dt = \frac{1}{k} \log(1+k) - \frac{1}{k} e^{-kx} E_1(x) + \frac{1}{k} E_1[(1+k)x], \quad (k>0); \quad (49)$$

$$\int_0^\infty e^{-kx} E_1(t) dt = \frac{1}{k} \log(1+k), \quad (k>0); \quad (50)$$

$$\int_0^x e^{kt} E_1(t) dt = -\frac{1}{k} \log(1-k) + \frac{1}{k} e^{kx} E_1(x) - \frac{1}{k} E_1[(1-k)x], \quad (1 > k > 0); \quad (51)$$

$$\int_0^\infty e^{kt} E_1(t) dt = -\frac{1}{k} \log(1+k) \quad (1 > k > 0); \quad (52)$$

$$\begin{aligned} \int_0^x e^{-kt} E_2(t) dt &= \frac{1}{k} \left\{ 1 - \frac{1}{k} \log(1+k) - e^{-kx} E_2(x) + \right. \\ &\quad \left. + \frac{1}{k} e^{-kx} E_1(x) - \frac{1}{k^2} E_1[(1+k)x] \right\}, \quad (k > 0); \end{aligned} \quad (53)$$

$$\int_0^\infty e^{-kt} E_2(t) dt = \frac{1}{k} \left\{ 1 - \frac{1}{k} \log(1+k) \right\}, \quad (k > 0); \quad (54)$$

$$\begin{aligned} \int_0^x e^{kt} E_2(t) dt &= -\frac{1}{k} \left\{ 1 + \frac{1}{k} \log(1-k) - e^{kx} E_2(x) - \frac{1}{k} e^{kx} E_1(x) + \right. \\ &\quad \left. + \frac{1}{k^2} E_1[(1-k)x] \right\}, \quad (1 > k > 0); \end{aligned} \quad /10 \quad (55)$$

$$\int_0^\infty e^{kt} E_2(t) dt = -\frac{1}{k} \left\{ 1 + \frac{1}{k} \log(1+k) \right\}, \quad (1 > k > 0); \quad (56)$$

$$\begin{aligned} \int_0^a e^{kt} E_1(|x-t|) dt &= \frac{1}{k} e^{kx} \log \frac{1+k}{1-k} + \frac{1}{k} e^{kx} E_1[(1+k)x] - \\ &\quad - \frac{1}{k} E_1(x) + \frac{1}{k} e^{ka} E_1(a-x) - \frac{1}{k} e^{kx} E_1[(1-k)(a-x)], \quad (k < 1); \end{aligned} \quad (57)$$

$$\int_{-\infty}^{+\infty} e^{kt} E_1(|t|) dt = \frac{1}{k} \log \frac{1+k}{1-k}, \quad (|k| < 1); \quad (58)$$

$$\int_0^x E_1^2(t) dt = 2 \log 2 + x E_1^2(x) + 2 E_1(2x) - 2 e^{-x} E_1(x); \quad (59)$$

$$\int_0^\infty \frac{e^{-kt}}{x+t} dt = e^{kx} E_1(kx). \quad (60)$$

6. Computing Practice for Tables of Functions $E_n(x)$

The Tables of functions $E_1(x)$ were calculated by means of eq.(8), and the power series on the right side of this formula was terminated at the term in x^{10} . Thus, the calculation was based on the formula

$$\begin{aligned} E_1(x) \approx & -c + |\log x| + x - \frac{1}{4} x^2 + \frac{1}{18} x^3 - \frac{1}{96} x^4 + \\ & + \frac{1}{600} x^5 - \frac{1}{4320} x^6 + \frac{1}{35280} x^7 - \frac{1}{322560} x^8 + \frac{1}{3265920} x^9 - \frac{1}{36288000} x^{10}. \end{aligned}$$

Remembering that the residual term of an alternating series is smaller in absolute value than the first rejected term, it can be established that the error of calculation by the above formula will be less than

$$\frac{x^{11}}{11 \cdot 11!} = \frac{x^{11}}{439084800} = 0.00000000227 x^{11}.$$

In our Tables, the greatest value of x is 0.6. Thus, $x^{11} = (0.6)^{11} = 0.00362797056$. Consequently, the residual term of the power series, in the most unfavorable case, will not exceed

$$0.00000000227 \times 0.00363 = 0.0000000000824,$$

i.e., one unit in the 11th place. Since computation by the above formula will involve an error due to rounding off the separate summands of this formula (except for the summands x and $-1/4 x^2$, which take their exact values), we may consider that in calculating with ten significant digits, the error of the final result will nowhere exceed half of the ninth digit. If the maximum value of x is 1.00, a similar calculation shows that we can everywhere be sure of the eighth digit. The functions $E_2(x)$, $E_3(x)$, and $E_4(x)$ were calculated by the aid of eqs.(22), (23), and (24) on the basis of the Tables constructed for the function $E_1(x)$.

It can be readily demonstrated that by using, as the basis of our calculation, a Table of this function with eight significant digits and a Table of the function e^{-x} with ten significant digits, we also will have eight significant digits for all values of $x \leq 1$ in the new Tables.

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II. SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

/12

Ye.S.Kuznetsov

The exact mathematical treatment of the problem of the scattering of light in the atmosphere reduces to the solution of a certain linear integral equation with finite limits. Theoretically, integral equations of this type may be solved by the method of successive approximation. In practice, however, the calculation of the successive approximations does not yield the desired result since, unless the successive approximations converge rapidly, a very large number of such approximations must be made to bring the approximate numerical solution close enough to the exact solution of the integral equation. Fortunately, in problems of atmospheric optics, good convergence of the successive approximations is ensured if the optical thickness τ^* of the atmosphere is small in value, ranging from 0.1 to 0.7. The quantity τ^* represents the upper limit of the integral entering into the integral equation, and therefore determines the rapidity of convergence of the successive approximations.

In this paper, we will consider the elementary problem of the theory of light scattering in the atmosphere, starting from the following assumptions:

- 1) The scattering indicatrix is spherical;
- 2) The albedo of the earth's surface is zero;
- 3) The light is incident on the upper boundary of the atmosphere in the form of a parallel beam making an angle of ζ with the vertical.

The problem formulated in this manner reduces to the solution of the integral equation

$$\varphi(\tau) = \frac{1}{4} e^{-(\tau^* - \tau) \sec \zeta} + \frac{1}{2} \int_0^{\tau^*} \varphi(t) E_1(|\tau - t|) dt. \quad (1)$$

Our problem consists in a detailed justification of the numerical solution of this integral equation by the method of successive approximation. Below, we give the answers to the following three questions: 1) evaluation of the error of the n-th approximation; 2) method of calculating the approximation of a given order from the preceding approximation; 3) evaluation of the error of approximate calculation of the integral resulting from the calculation of the successive approximations.

1. Evaluation of the Error of Approximation of a Given Order

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The method of successive approximations, as applied to the integral equation (1), consists in the selection of the arbitrary function $\varphi_0(\tau)$ as the

zero-th approximation, followed by a determination of the $(n+1)$ -th approximation based on the formula

$$\varphi_{n+1}(\tau) = \frac{1}{4} e^{-(\tau^* - \tau) \sec \xi} + \frac{1}{4} \int_0^{\tau^*} \varphi_n(t) E_1(|\tau - t|) dt, \quad (2)$$

$$n = 1, 2, 3, \dots$$

It can be shown that the sequence of the functions $\varphi_n(\tau)$ is uniformly convergent to the solution of the integral equation (1), and that $\varphi(\tau) = \lim_{n \rightarrow \infty} \varphi_n(\tau)$ is the unique solution of the problem. The sequence $\varphi_n(\tau)$ will converge more rapidly for $\varphi(\tau)$ than a geometric progression with the denominator $1 - E_2\left(\frac{\tau^*}{2}\right)$. The numerical values of this expression for various values of τ^* are as follows:

τ^*	0.2	0.3	0.4	0.5	0.6
$1 - E_2\left(\frac{\tau^*}{2}\right)$	0.28	0.36	0.43	0.48	0.53

In practice, the calculations can be carried only to a certain approximation $\varphi_n(\tau)$, where n is not too great an integer; this raises the question as to the error due to the replacement of the exact solution by the n -th approximation $\varphi_n(\tau)$. Let R_n be this error. The exact solution of the integral equation can be represented in the form of the infinite series

$$\begin{aligned} \varphi(\tau) &= \varphi_0(\tau) + [\varphi_1(\tau) - \varphi_0(\tau)] + [\varphi_2(\tau) - \varphi_1(\tau)] + \dots + \\ &\quad + [\varphi_n(\tau) - \varphi_{n-1}(\tau)] + \dots \end{aligned} \quad (3)$$

or

$$\begin{aligned} \varphi(\tau) &= \varphi_0(\tau) + [\varphi_1(\tau) - \varphi_0(\tau)] + [\varphi_2(\tau) - \varphi_1(\tau)] + \dots + \\ &\quad + [\varphi_n(\tau) - \varphi_{n-1}(\tau)] + R_n, \end{aligned} \quad (4)$$

where

$$R_n = [\varphi_{n+1}(\tau) - \varphi_n(\tau)] + [\varphi_{n+2}(\tau) - \varphi_{n+1}(\tau)] + \dots \quad (5)$$

or

$$\varphi(\tau) = \varphi_n(\tau) + R_n(\tau). \quad (6)$$

Consider the two approximations $\varphi_v(\tau)$ and $\varphi_{v+1}(\tau)$, where $v \geq n$. As follows from eq.(2),

$$\varphi_{v+1}(\tau) - \varphi_v(\tau) = \frac{1}{2} \int_0^{\tau^*} [\varphi_v(t) - \varphi_{v-1}(t)] E_1(|\tau - t|) dt.$$

Hence

$$|\varphi_{v+1}(\tau) - \varphi_v(\tau)| \leq \frac{1}{2} \int_0^{\tau^*} |\varphi_v(t) - \varphi_{v-1}(t)| E_1(|\tau - t|) dt$$

and

$$|\varphi_{v+1}(\tau) - \varphi_v(\tau)| < \frac{1}{2} \max |\varphi_v(\tau) - \varphi_{v-1}(\tau)| \int_0^{\tau^*} E_1(|\tau - t|) dt.$$

However,

$$\frac{1}{2} \int_0^{\tau^*} E_1(|\tau - t|) dt = 1 - \frac{E_2(\tau) + E_2(\tau^* - \tau)}{2} < q = 1 - E_2\left(\frac{\tau^*}{2}\right).$$

Consequently,

$$|\varphi_{v+1}(\tau) - \varphi_v(\tau)| < q \max |\varphi_v(\tau) - \varphi_{v-1}(\tau)|.$$

Let us put

$$\max |\varphi_n(\tau) - \varphi_{n-1}(\tau)| = D_n.$$

Hence, according to eq.(7),

$$|\varphi_{n+1}(\tau) - \varphi_n(\tau)| < qD_n$$

and, in particular,

But then

$$\max |\varphi_{n+1}(\tau) - \varphi_n(\tau)| < qD_n.$$

and

$$|\varphi_{n+2}(\tau) - \varphi_{n+1}(\tau)| < q^2 D_n$$

In general,

$$\max |\varphi_{n+2}(\tau) - \varphi_{n+1}(\tau)| < q^2 D_n.$$

(8)

$$\max |\varphi_{v+1}(\tau) - \varphi_v(\tau)| < q^{v-n+1} D_n.$$

From eq.(5), we have

$$|R_n| < \sum_{v=n}^{\infty} |\varphi_{v+1}(\tau) - \varphi_v(\tau)|.$$

On the basis of eq.(8) we thus obtain

$$|R_n| < D_n \sum_{v=n}^{\infty} q^{v-n+1} = qD_n \sum_{v=n}^{\infty} q^{v-n}$$

or

$$|R_n| < \frac{qD_n}{1-q}, \quad (9)$$

which likewise can be represented in the form of

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$$|R_n| < D_n \frac{1 - E_2\left(\frac{\tau^*}{2}\right)}{E_2\left(\frac{\tau^*}{2}\right)}. \quad (10)$$

The resultant inequality permits correlating the evaluation of the error R_n with the value of the last calculated difference between two approximations. It remains to discover the influence on this evaluation exerted by the factor

$$\frac{1 - E_2\left(\frac{\tau^*}{2}\right)}{E_2\left(\frac{\tau^*}{2}\right)}.$$

The following Table gives the values of this factor for various values of τ^* :

τ^*	0.2	0.3	0.4	0.5	0.6
$\frac{1 - E_2\left(\frac{\tau^*}{2}\right)}{E_2\left(\frac{\tau^*}{2}\right)}$	0.38	0.56	0.74	0.93	1.13

Thus, within the limits of the values of τ^* of interest here, the following evaluation can be used:

$$|R_n| < D_n. \quad (11)$$

In other words, the error of the n -th approximation will be less than the maximum value of the difference between the last two calculated approximations. If a less coarse evaluation is required, we can always use the inequality (10) or the above small Table. We note that in problems of atmospheric optics, even in the case of a highly turbid atmosphere, it is never possible to get double the value of D_n . This doubling is attained approximately at $\tau^* = 1$; in this case

$$\frac{1 - E_2\left(\frac{\tau^*}{2}\right)}{E_2\left(\frac{\tau^*}{2}\right)} = 2.03.$$

2. Technique of Calculating the Successive Approximations

It follows from the general equation (2) for calculating the $(n+1)$ -th approximation from the n -th approximation that the problem in all cases reduces to the calculation of an integral of the following form:

$$I(\tau) = \int_0^{\tau} f(t) E_1(|\tau - t|) dt, \quad (12)$$

where $f(\tau)$ is an assigned function. The ordinary methods of mechanical quadrature encounter difficulties in this case, owing to the fact that the function $E_1(|\tau - t|)$ becomes infinite at $\tau = t$. This difficulty can be avoided, as demonstrated below, by selecting as the dividing points in the numerical integration those values of τ for which the integral (12) is calculated. 16

Let us divide the integral $(0, \tau^*)$ into n equal parts, each of length h . Then $\tau^* = nh$, and

$$I(\tau) = \sum_{m=0}^{n-1} \int_{mh}^{(m+1)h} f(t) E_1(|\tau - t|) dt = \sum_{m=0}^{n-1} I_m^{n+1}(\tau).$$

Let

$$\begin{aligned} \tau &= kh \quad (k = 0, 1, 2, \dots, n), \\ t &= mh \quad (m = 0, 1, 2, \dots, n). \end{aligned}$$

In our calculations we used:

$$\begin{aligned} k &= 0.01 \quad (\text{at } \tau^* = 0.2; 0.3), \\ h &= 0.02 \quad (\text{at } \tau^* = 0.4; 0.5; 0.6). \end{aligned}$$

Thus, we perform the calculation of our integral only at equidistant values of the argument, spaced by 0.01 (or 0.02). Consequently,

$$I(kh) = \sum_{m=0}^{n-1} \int_{mh}^{(m+1)h} f(t) E_1(|kh - t|) dt$$

or

$$I(kh) = \sum_{m=0}^{k-1} \int_{mh}^{(m+1)h} f(t) E_1(kh - t) dt + \sum_{m=k}^{n-1} \int_{mh}^{(m+1)h} f(t) E_1(t - kh) dt. \quad (13)$$

In the integrals, the first sum $m \leq k - 1$. Consider one of the integrals satisfying this condition

$$I_m^{n+1}(k) = \int_{mh}^{(m+1)h} f(t) E_1(kh - t) dt$$

and let us put, approximately

$$f(t) = \alpha_m + \beta_m t \quad (mh \leq t \leq (m+1)h).$$

Substituting this expression into our integral and performing the appropriate transformations, we arrive at the following expression:

$$\begin{aligned} I_m^{m+1}(k) &= (\alpha_m + \beta_m t) \{E_2[(k-m-1)h] - E_2[(k-m)h]\} - \\ &- \beta \{[E_3[(k-m-1)h] - E_3[(k-m)h]] + (k-m-1)hE_2[(k-m-1)h] - \\ &- (k-m)hE_2[(k-m)h]\}. \end{aligned}$$

As a result of the integration, the singularity which we had shifted to l7 the end of the integral has now vanished. The values of α_m and β_m are readily calculated from the assigned values of the function $f(\tau)$. In fact,

$$\begin{aligned} f(mh) &= \alpha_m + \beta_m mh, \\ f[(m+1)h] &= \alpha_m + \beta_m (m+1)h. \end{aligned}$$

Hence,

$$\beta_m = \frac{f[(m+1)h] - f(mh)}{h} = \frac{\Delta f(mh)}{h},$$

$$\alpha_m = f(mh) - m\Delta f(mh).$$

Substituting these expressions for α_m and β_m in the above formula, we finally obtain

$$I_m^{m+1}(k) = P_{km} f(nh) + Q_{km} \frac{\Delta f(mh)}{h} \quad (m \leq k-1), \quad (14)$$

where

$$\left. \begin{aligned} P_{km} &= E_2[(k-m-1)h] - E_2[(k-m)h]; \\ Q_{km} &= -\{E_3[(k-m-1)h] - E_3[(k-m)h] - hE_2[(k-m-1)h]\}. \end{aligned} \right\} \quad (15)$$

Similarly,

$$I_m^{m+1}(k) = P_{km} f(nh) + Q_{km} \frac{\Delta f(mh)}{h} \quad (m \geq k), \quad (16)$$

where

$$\left. \begin{aligned} P_{km} &= E_2[(m-k)h] - E_2[(m-k+1)h], \\ Q_{km} &= E_3[(m-k)h] - E_3[(m-k+1)h] - hE_2[(m-k+1)h]. \end{aligned} \right\} \quad (17)$$

Substituting the expressions (14) and (15) into eq.(13), we obtain, for the integral $I(k)$ sought, the following final approximate expression:

$$I(k) = \sum_{m=0}^{n-1} \left[P_{km} f(mh) + Q_{km} \frac{\Delta f(mh)}{h} \right], \quad (18)$$

where the factors $P_{k\bullet}$ and $Q_{k\bullet}$ are calculated by means of eqs.(15) for $m \leq k - 1$, and from eqs.(17) for $m \geq k$. The coefficients $P_{k\bullet}$ and $Q_{k\bullet}$ are independent of the function $f(t)$ and may be calculated once and for all for an assigned h . The construction of Tables of these coefficients is substantially simplified by using the following properties of the symbols $P_{k\bullet}$ and $Q_{k\bullet}$, which can be proved without trouble:

- 1) $P_{r,r+s} = F(s)$ (independent of r) ;
- 2) $P_{r,r+s} = P_{r,r-s-1}$;
- 3) $Q_{r,r+s} = F(s)$ (independent of r) ;
- 4) $Q_{r,r+s} = -Q_{r,r-s-1} + h[E_2(sh) - E_2((s+1)h)]$.

These properties permit calculation of the values of $P_{k\bullet}$ only for the case $k = 0$. Having calculated this column, we then rewrite the values without change in the next column but shifted one row lower, and so on. The remaining unfilled spaces above and to the right of the diagonal of the Table are filled with the digits of the same columns, "reflected" relative to the diagonal. As a result we obtain the following form of Table for the coefficients $P_{k\bullet}$:

$m \backslash k$	0	1	2	3	4	5	...
0	P_{00}	P_{00}	P_{01}	P_{02}	P_{03}	P_{04}	...
1	P_{01}	P_{00}	P_{00}	P_{01}	P_{02}	P_{03}	...
2	P_{02}	P_{01}	P_{00}	P_{00}	P_{01}	P_{02}	...
3	P_{03}	P_{02}	P_{01}	P_{00}	P_{00}	P_{01}	...
4	P_{04}	P_{03}	P_{02}	P_{01}	P_{00}	P_{00}	...
5	P_{05}	P_{04}	P_{03}	P_{02}	P_{01}	P_{00}	...
...

The Table for the $Q_{k\bullet}$ is constructed in the same way, but instead of "reflection" relative to the diagonal, a more complex law must be applied here. Schematically this Table can be represented in the following form, where the quantities q_s represent the expression

$$h[E_2(sh) - E_2((s+1)h)].$$

To calculate the integral $I(k)$ for an assigned value of k , it is sufficient to multiply the values of $f(mh)$ by the digits of the column of the number k of the first Table, and the values of $\frac{\Delta f(mh)}{h}$ by the digits of the

column of the same number of the second Table, after which all the products so obtained are summated by eq.(18). Tables of the coefficients $P_{0\bullet}$, $Q_{0\bullet}$ and $Q_{n\bullet}$ are given as appendices to this paper. It follows from the above that the

values of P_{km} and Q_{km} may be found for any values of k from these three Tables.

$m \backslash k$	0	1	2	3	4	5	...
0	Q_{00}	$-Q_{00} - q_0$	$-Q_{01} - q_1$	$-Q_{02} - q_2$	$-Q_{03} - q_3$	$-Q_{04} + q_4$...
1	Q_{01}	Q_{00}	$-Q_{00} + q_0$	$-Q_{01} - q_1$	$-Q_{02} - q_2$	$-Q_{03} + q_3$...
2	Q_{02}	Q_{01}	Q_{00}	$-Q_{00} + q_0$	$-Q_{01} - q_1$	$-Q_{02} + q_2$...
3	Q_{03}	Q_{02}	Q_{01}	Q_{00}	$-Q_{00} + q_0$	$-Q_{01} + q_1$...
4	Q_{04}	Q_{03}	Q_{02}	Q_{01}	Q_{00}
5	Q_{05}	Q_{04}	Q_{03}	Q_{02}	Q_{01}	Q_{00}	...
...

Equation (18) also can be transformed to the form of

$$I(k) = \sum_{m=0}^n A_{km} f(mh),$$

where

$$A_{k0} = P_{k0} - \frac{Q_{k0}}{h}, \quad A_{km} = P_{km} - \frac{Q_{k,m-1} - Q_{k,m-2}}{h}, \quad (m=1, 2, \dots, n-1),$$

$$A_{kn} = \frac{Q_{k,n-1}}{h}.$$

This formula reduces the volume of calculations compared with eq.(18). /19
The matrix A_{km} , however, is less convenient than the matrices P_{km} and Q_{km} .

3. Evaluation of the Error of Quadrature

The calculation of the integral $I(\tau)$ was reduced to the calculation of n integrals of the form:

$$I_m^{m+1}(k) = \int_{mh}^{(m+1)h} f(t) E_1(kh-t) dt \quad (m \leq k-1) \quad (19)$$

and

$$I_m^{m+1}(k) = \int_{mh}^{(m+1)h} f(t) E_1(t-kh) dt \quad (m \geq k). \quad (20)$$

Let us represent the function $f(t)$ in the interval (a_1, a_2) by the aid of the Lagrange interpolation formula corresponding to the case of linear interpolation:

$$f(t) = \frac{t-a_2}{a_1-a_2} f(a_1) + \frac{t-a_1}{a_2-a_1} f(a_2) + \frac{(t-a_1)(t-a_2)}{1 \cdot 2} f''(\xi), \quad (21)$$

where $a_1 < \xi < a_2$.

In our case, we must put

$$a_1 = mh, \quad a_2 = (m+1)h.$$

Then eq.(21) takes the following form:

$$\begin{aligned} f(t) = & -\frac{t-(m+1)h}{h} f(mh) + \frac{t-mh}{h} f[(m+1)h] + \\ & + \frac{(t-mh)[t-(m+1)h]}{1 \cdot 2} f''(\xi_m), \end{aligned} \quad (22)$$

where $mh < \xi_m < (m+1)h$.

Equation (22) is easily put into the form of

$$f(t) = \alpha_m + \beta_m t + \frac{(t-mh)[t-(m+1)h]}{1 \cdot 2} f''(\xi_m), \quad (23)$$

where α_m and β_m are calculated by the formulas derived earlier:

$$\begin{aligned} \alpha_m &= f(mh) - m\Delta f(mh), \\ \beta_m &= \frac{\Delta f(mh)}{h}. \end{aligned}$$

Substituting eq.(23) into eq.(19) or (20), we obtain the integral $I_m^{k+1}(k)$ in 20 the form of two summands, the first of which coincides exactly with the expression studied above, while the second gives the error of quadrature. Denoting this error by R_{km} , we obtain:

$$\begin{aligned} R_{km} &= \frac{1}{2} \int_{mh}^{(m+1)h} [t-mh][t-(m+1)h] E_1(kh-t) f''(\xi_m) dt \quad (m \leq k-1); \\ R_{km} &= \frac{1}{2} \int_{mh}^{(m+1)h} [t-mh][t-(m+1)h] E_1(t-kh) f''(\xi_m) dt \quad (m \geq k). \end{aligned}$$

Since $mh \leq t \leq (m+1)h$, the product $[t-mh] \cdot [t-(m+1)h]$ retains the minus sign in the integration interval. Therefore, using the theorem of the mean value of an integral, we can write the preceding expression in the form of

$$\left. \begin{aligned} R_{km} &= \frac{1}{2} f''(\eta_m) \int_{mh}^{(m+1)h} [t-mh][t-(m+1)h] E_1(kh-t) dt \\ R_{km} &= \frac{1}{2} f''(\eta_m) \int_{mh}^{(m+1)h} [t-mh][t-(m+1)h] E_1(t-kh) dt \end{aligned} \right\} \begin{array}{l} (m \leq k-1); \\ (m \geq k), \end{array} \quad (24)$$

where $mh < \eta_m < (m+1)h$. Putting

$$U_{km} = \frac{1}{2} \int_{mh}^{(m+1)h} [t - mh][t - (m+1)h] E_1(kh - t) dt \quad (m \leq k-1), \quad (25)$$

$$U_{km} = \frac{1}{2} \int_{mh}^{(m+1)h} [t - mh][t - (m+1)h] E_1(t - kh) dt \quad (m \geq k), \quad (26)$$

the equations of the system (24) can be combined into a single formula:

$$R_{km} = f''(\eta_m) U_{km} \quad (mh < \eta_m < (m+1)h). \quad (27)$$

Consider the case $m \leq k-1$. Let us put, in eq.(25),

$$kh - t = x.$$

Then,

$$U_{km} = \frac{1}{2} \int_{(k-m-1)h}^{(k-m)h} [(k-m)h - x][(k-m-1)h - x] E_1(x) dx \quad (m \leq k-1)$$

or

$$U_{km} = \frac{1}{2} \int_{(k-m-1)h}^{(k-m)h} \{x^2 - [2(k-m)-1]hx + (k-m)(k-m-1)h^2\} E_1(x) dx.$$

Calculation yields:

$$\begin{aligned} U_{km} = & E_4[(k-m-1)h] - E_4[(k-m)h] - \\ & - \frac{h}{2} \{E_3[(k-m-1)h] + E_3[(k-m)h]\} \quad (m \leq k-1). \end{aligned} \quad (28)$$

In eq.(26) we must put:

$$t - kh = x,$$

which brings us to the expression

$$U_{km} = \frac{1}{2} \int_{(m-k)h}^{(m-k-1)h} [x - (m-k)h][x - (m-k-1)h] E_1(x) dx \quad (m \geq k)$$

or

$$U_{km} = \frac{1}{2} \int_{(m-k)h}^{(m-k-1)h} \{x^2 - [2(m-k)+1]hx + (m-k)(m-k+1)h^2\} E_1(x) dx.$$

In this case, we will have

$$U_{km} = E_4[(m-k)h] - E_4[(m-k+1)h] - \frac{h^2}{2} \{E_3[(m-k)h] + E_3[(m-k+1)h]\} \quad (m \geq k). \quad (29)$$

We obtain the following expression for the total error:

$$R_k = \sum_{m=0}^{n-1} f''(\eta_m) U_{km}, \quad (30)$$

where U_{km} is calculated by means of eqs.(28) and (29), and

$$0 < \eta_0 < h < \eta_1 < 2h < \eta_2 < 3h < \dots < mh < \eta_m < (m+1)h < \dots \\ \dots < (m-1)h < \eta_{m-1} < \tau.$$

The function U_{km} entering into the expression of error [eq.(30)] satisfies the following conditions:

- (1) $U_{km} < 0$,
- (2) $U_{r,r+s} = F(s)$ (independent of r),
- (3) $U_{r,r+s} = U_{r,r-s-1}$,
- (4) $\frac{\partial U_{km}}{\partial m} > 0 \quad (m \geq k), \quad \frac{\partial U_{km}}{\partial m} < 0 \quad (m \leq k-1)$.

The properties (2) and (3) show that the quantity U_{km} obeys the same laws as P_{km} , so that analogous Tables can be compiled for it. We will discuss the proof of property (4) in greater detail. /22

Differentiation of eq.(26) with respect to m gives, for example,

$$\frac{\partial U_{km}}{\partial m} = \frac{h}{2} \int_{m^2}^{(m+1)h} [(2m+1)h - 2t] E_1(t - kh) dt.$$

Hence, making use of the substitution $t = (m + \frac{1}{2})h + t'$, we obtain:

$$\begin{aligned} \frac{\partial U_{km}}{\partial m} &= -h \int_{-h/2}^{h/2} t' E_1 \left[\left(m - k + \frac{1}{2} \right) h + t' \right] dt' = \\ &= h \int_0^{h/2} t' \left\{ E_1 \left[\left(m - k + \frac{1}{2} \right) h - t' \right] - \right. \\ &\quad \left. - E_1 \left[\left(m - k + \frac{1}{2} \right) h + t' \right] \right\} dt' > 0. \end{aligned}$$

For $h > 0$ and $m > k$, the right-hand side of this equality is positive, so that we obtain property (4).

Thus, $|U_{kk}|$ is a decreasing function of m . We obtain its greatest value at $m = k$. Setting $m = k$ in eq.(29), we obtain this value, greatest in absolute magnitude, in the form of

$$U_{kk} = \frac{1}{3} - E_4(h) - \frac{h}{2} \left[\frac{1}{2} - E_3(h) \right].$$

This yields the following evaluation of the error R_k :

$$|R_k| < |U_{kk}| \sum_{m=0}^{n-1} |f''(\eta_m)|$$

or

$$|R_k| < n |U_{kk}| |f''(\eta)|, \quad (31)$$

where

$$0 < \eta < \tau^*.$$

The quantity U_{kk} is very small in absolute value. Thus, for $h = 0.01$, we obtain:

$$U_{kk} = -0.000000406.$$

Consequently, in this case

$$|R_k| < 0.000000406 n |f''(\eta)|.$$

As an example, consider the calculation of the first approximation when the free term of the integrated equation (1) is taken as the zero-th approximation: /23

$$\frac{1}{4} e^{-(\tau^* - \tau) \sec \zeta}.$$

Then, we must calculate the integral (12) for $f(\tau) = \frac{1}{4} e^{-(\tau^* - \tau) \sec \zeta}$. The second derivative of this function is

$$f''(\tau) = \frac{\sec^2 \zeta}{4} e^{-(\tau^* - \tau) \sec \zeta}.$$

Obviously,

$$f''(\tau) < \frac{\sec^2 \zeta}{4}.$$

The greatest value of ζ in our calculations is 76° . Since $\sec 76^\circ = 3.864$, it follows that

$$f''(\tau) < 3.732 < 4$$

for any τ . For instance, let us take the value 30 for n . Then,

$$|R_k| < 0.000048 < 0.00005.$$

In reality, in view of the roughness of the estimates, we would expect even smaller errors.

4. Allowance for the Albedo of the Earth's Surface

Equation (1) relates to the case when the earth's surface is regarded as a black body. Describing reflection of light from the earth's surface by the aid of the albedo, the problem reduces to the solution of the integral equation:

$$\begin{aligned} \varphi_q(\tau) = & \frac{1}{4} e^{-\sec \zeta (\tau^* - \tau)} + \frac{1}{2} q \cos \zeta e^{-\tau^* \sec \zeta} E_2(\tau) + \\ & + \int_0^{\tau^*} \varphi_q(t) \left[\frac{1}{2} E_1(|\tau - t|) + q E_2(\tau) E_2(t) \right] dt, \end{aligned} \quad (32)$$

where q is the albedo. In the case $q = 0$ we return to eq.(1), which can now be written as follows:

$$\varphi_0(\tau) = \frac{1}{4} e^{-\sec \zeta (\tau^* - \tau)} + \frac{1}{2} \int_0^{\tau^*} \varphi_0(t) E_1(|\tau - t|) dt. \quad (33)$$

Let us introduce the auxiliary integral equation:

$$\omega(\tau) = \frac{1}{2} E_2(\tau) + \frac{1}{2} \int_0^{\tau^*} \omega(t) E_1(|\tau - t|) dt. \quad (34)$$

It can be shown that the solution of eq.(32) $\varphi_q(\tau)$ is represented in terms of the solutions of eqs.(33) and (34), so that $\varphi_0(\tau)$ and $\omega(\tau)$ can be determined by the formula

$$\varphi_q(\tau) = \varphi_0(\tau) + H \omega(\tau), \quad (35)$$

where

$$H = \frac{q \left(\cos \zeta e^{-\tau^* \sec \zeta} + 2 \int_0^{\tau^*} \varphi_0(t) E_2(t) dt \right)}{1 - 2q \int_0^{\tau^*} \omega(t) E_2(t) dt}. \quad (36)$$

The solution of eq.(34) can be obtained by means of the numerical method described in the preceding Sections. Thus, it is no longer necessary to solve eq.(32) directly for each albedo, which greatly reduces the amount of computational work.

5. Selection of the Zero-th Approximation

It is well known that the proper selection of the zero-th approximation may considerably decrease the amount of computational work in the numerical solution of integral equations. In the case of eqs.(33) and (34), the following possibilities exist here.

- 1) The free term of eq.(33)

$$\tilde{\phi}(\tau) = \frac{1}{4} e^{-\sec \zeta (\tau^* - \tau)}. \quad (37)$$

can be used as the zero-th approximation. This choice of the zero-th approximation is convenient from the physical viewpoint since, in this case, the individual approximations exactly correspond to light scattering of the various multiplicities. In particular, the zero-th approximation is itself equivalent to taking account of the light scattering of the first order alone. From the computational viewpoint, however, this form of the zero-th approximation is disadvantageous since it is too far from the true solution of the integral equation.

- 2) Any approximate solution of the problem may be taken as the zero-th approximation, for instance, the solution that can be obtained by the Schwarzschild method.

At zero albedo of the earth's surface, this approximate solution has the following form:

$$\begin{aligned} \tilde{\phi}(\tau) &= \left(\frac{1}{2} + \cos \zeta \right) \left[\cos \zeta + \left(\frac{1}{2} - \cos \zeta \right) e^{-(\tau^* - \tau) \sec \zeta} \right] - \\ &- \cos \zeta \left[\left(\frac{1}{2} + \cos \zeta \right) + \left(\frac{1}{2} - \cos \zeta \right) e^{-(\tau^* - \tau) \sec \zeta} \right] \frac{\tau^* - \tau + \frac{1}{2}}{\tau^* + 1}. \end{aligned} \quad (38)$$

Actually, it was this expression which we selected as the zero-th approximation in all our calculations. /25

The zero-th approximation of the Schwarzschild type for the auxiliary integral equation (34) reads

$$\tilde{w}(\tau) = \frac{\tau^* - \tau + \frac{1}{2}}{\tau^* + 1}.$$

However, a more accurate result can be obtained by calculating the following approximation and taking it as the zero-th, i.e., by putting

$$\tilde{\omega}(\tau) = \frac{1}{2} E_2(\tau) + \frac{1}{2} \int_0^{\tau^*} \frac{\tau^* - t + \frac{1}{2}}{\tau^* + 1} E_1(|\tau - t|) dt.$$

Calculation then yields

$$\tilde{\omega}(\tau) = \frac{\tau^* - \tau + \frac{1}{2} + \frac{1}{4} [E_2(\tau) - E_2(\tau^* - \tau)] - \frac{1}{2} [E_3(\tau) - E_3(\tau^* - \tau)]}{\tau^* + 1}. \quad (39)$$

We used this formula as basis in calculating the zero-th approximation for eq.(34).

Several other methods of refining the zero-th approximation are discussed in B.V.Ovchinskiy's papers.

III. IMPROVEMENT OF THE ZERO-TH APPROXIMATION

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B.V.Ovchinskiy

The question of selection of the zero-th approximation becomes acute in using the method of successive approximations for the numerical solution of the fundamental integral equation of the theory of light scattering in the atmosphere:

$$\varphi(\tau) = \frac{1}{4} e^{-(\tau^* - \tau) \sec \zeta} + \frac{1}{2} \int_0^{\tau^*} \tilde{\varphi}(t) E_1(|\tau - t|) dt \quad (1)$$

The proper selection of the zero-th approximation determines the amount of additional approximations needed for obtaining a solution of the required accuracy.

Of course, the free term of eq.(1) can be selected as the zero-th approximation.

From the viewpoint of computational technique, however, this cannot be regarded as a good choice since the value of the free term differs too greatly from the exact solution of the integral equation, leading to the necessity of a large number of approximations to obtain a sufficiently exact result.

For this reason, the solution obtainable by the Schwarzschild method was adopted as the initial approximation. This was done in constructing the solution of eq.(1) in the case of $\tau^* = 0.2, 0.3$, and $\zeta = 30, 45, 60$, and 76° . For other values of τ^* (0.4; 0.5; 0.6), the following method of improving the zero-th approximation was adopted:

The zero-th approximation is considered in the form

$$\varphi_0 = \tilde{\varphi} + A + B\tau + C\tau^2, \quad (2)$$

where $\tilde{\varphi}$ is the solution by the Schwarzschild method while A, B, and C are constants to be determined.

To determine these constants, let us substitute φ in eq.(1) by the expression for φ_0 according to eq.(2):

$$\begin{aligned} \tilde{\varphi} + A + B\tau + C\tau^2 &= \frac{1}{4} e^{-(\tau^* - \tau) \sec \zeta} + \frac{1}{2} \int_0^{\tau^*} \tilde{\varphi}(t) E_1(|\tau - t|) dt + \\ &+ \frac{1}{2} A \int_0^{\tau^*} E_1(|\tau - t|) dt + \frac{1}{2} B \int_0^{\tau^*} t E_1(|\tau - t|) dt + \frac{1}{2} C \int_0^{\tau^*} t^2 E_1(|\tau - t|) dt. \end{aligned} \quad (3)$$

The integrals entering into eq.(3) can be calculated on the basis of eqs.(I,43),
(I,44), and (I,45) of the present paper:

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$$\begin{aligned}\int_0^{\tau^*} E_1(|\tau - t|) dt &= 2 - E_2(\tau) - E_2(\tau^* - \tau); \\ \int_0^{\tau^*} t E_1(|\tau - t|) dt &= 2\tau + E_3(\tau) - \tau^* E_2(\tau^* - \tau) - E_3(\tau^* - \tau); \\ \int_0^{\tau^*} t^2 E_1(|\tau - t|) dt &= 2\tau^2 - \tau^{*2} E_2(\tau^* - \tau) - 2\tau^* E_3(\tau^* - \tau) - \\ &\quad - 2E_4(\tau^* - \tau) - 2E_4(\tau) + \frac{4}{3}.\end{aligned}$$

The following equation is obtained from these formulas for determining A, B, and C:

$$[E_2(\tau) + E_2(\tau^* - \tau)]A + [\tau^* E_2(\tau^* - \tau) + E_3(\tau^* - \tau) - E_3(\tau)]B + \\ + \left[\tau^{*2} E_2(\tau^* - \tau) + 2\tau^* E_3(\tau^* - \tau) + 2E_4(\tau^* - \tau) + 2E_4(\tau) - \frac{4}{3} \right]C = 2(\varphi_1 - \tilde{\varphi}) \quad (4)$$

where

$$\varphi_1 = -\frac{1}{4} e^{-(\tau^* - \tau) \sec \zeta} + \frac{1}{2} \int_0^{\tau^*} \tilde{\varphi}(t) E_1(|\tau - t|) dt$$

is the first approximation of our equation, if the zero-th approximation is taken according to Schwarzschild. Equation (4) contains three unknowns. To find these unknowns, τ can be given all possible values. This yields a system of equations which, after solving it by the method of least squares, furnishes the approximate values of A, B, and C.

We used a different procedure: The quantity τ was given only three values, namely, $\tau = 0$, $\tau = \frac{\tau^*}{2}$, $\tau = \tau^*$, while A, B, and C were uniquely determined from the following system:

$$\left. \begin{aligned} [1 + E_2(\tau^*)]A + \left[\tau^* E_2(\tau^*) + E_3(\tau^*) - \frac{1}{2} \right]B + \left[\tau^{*2} E_2(\tau^*) + \right. \\ \left. + 2\tau^* E_3(\tau^*) + 2E_4(\tau^*) - \frac{2}{3} \right]C = 2[\varphi_1(0) - \tilde{\varphi}(0)]; \\ 2E_2\left(\frac{\tau^*}{2}\right)A + \tau^* E_2\left(\frac{\tau^*}{2}\right)B + \left[\tau^{*2} E_2\left(\frac{\tau^*}{2}\right) + 2\tau^* E_3\left(\frac{\tau^*}{2}\right) + \right. \\ \left. + 4E_4\left(\frac{\tau^*}{2}\right) - \frac{4}{3} \right]C = 2\left[\varphi_1\left(\frac{\tau^*}{2}\right) - \tilde{\varphi}\left(\frac{\tau^*}{2}\right)\right]; \\ [1 + E_2(\tau^*)]A + \left[\tau^* + \frac{1}{2} - E_3(\tau^*) \right]B + \left[\tau^{*2} + \tau^* + \right. \\ \left. + 2E_4(\tau^*) - \frac{2}{3} \right]C = 2[\varphi_1(\tau^*) - \tilde{\varphi}(\tau^*)]\end{aligned} \right\} \quad (5)$$

The quantities $\varphi_1(0)$, $\varphi_1\left(\frac{\tau^*}{2}\right)$, and $\varphi_1(\tau^*)$, entering into the right-hand /28 sides of eq.(5), were determined by numerical integration using the method described in the first paper.

As stated above, for $\tau^* = 0.2$ and $\tau^* = 0.3$, the values obtained by the Schwarzschild method were taken as the zero-th approximation, i.e., $\varphi_0(\tau)$ was taken as equal to $\tilde{\varphi}(\tau)$. To define the changes introduced by the addition of the parabolic trinomial $A + B\tau + C\tau^2$, we performed additional calculations for $\tau^* = 0.2$ and $\tau^* = 0.3$ and for all ζ . The graphs at the end of this book give the successive approximations, using φ , i.e., the solution obtained by the Schwarzschild method, as the zero-th approximation. The broken curve corresponds to the values of $\varphi_0 = \varphi + A + B\tau + C\tau^2$. These diagrams indicate that the broken curve corresponds to the second and third approximations. A somewhat poorer result is usually obtained for the terminals of the curve.

In view of the relatively small amount of work involved in calculating the parabolic trinomial $A + B\tau + C\tau^2$, which saves two or three approximations, the advantage of this method is obvious.

IV. COMPARISON OF THE APPROXIMATE SOLUTIONS OF THE PROBLEM
OF LIGHT SCATTERING IN THE ATMOSPHERE WITH THE SOLUTIONS
OBTAINED BY THE AID OF INTEGRAL EQUATIONS

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Ye.S.Kuznetsov

Until recently, the problem of the propagation of radiant energy in absorbing and scattering media was solved mainly by approximate methods and no exact data were available on the degree of deviation of these approximate solutions from the rigorous solution of the problem.

The approximate methods used in practical application can be subdivided into two groups.

The first group comprises the methods based on the direct calculation of single scattering and sometimes also of double scattering (below, we will refer only to problems of pure scattering). All approximate formulas relating to this class are readily obtained by using, in all relations of the exact theory, the free term of the integral equation of the theory of light scattering instead of the exact solution of that equation.

The second group includes the methods for solving the problem of light scattering based on the application of some form of approximate equation for the transfer of radiant energy. The approximate transfer equations established by Schwarzschild are the most widely used formulas in astrophysics and meteorology. The approximate method of solution proposed by Eddington for the transfer equations is less often employed. Finally, we should mention the Chandrasekhar method which is a generalized Schwarzschild method, permitting, in principle, solutions with any desired degree of exactness.

The numerical solutions obtained by us for the integral equation of the theory of light scattering in the atmosphere permit a comparison of the various approximate solutions with the exact solution (within the limits of error of the numerical solution of the integral equation).

We preface the description of the results of such a comparison by the derivation of the approximate solutions, based on the Schwarzschild and Eddington methods. Both these solutions have been derived anew in their application to the problem of light scattering in the earth's atmosphere, the basic difference in our formulation of the problem lying in the fact that we allowed for the albedo of the earth's surface, which was disregarded in the classical Schwarzschild and Eddington solutions.

The Chandrasekhar method will not be discussed here, since a special article by B.V.Ovchinskiy is devoted to it.

Confining the calculation to the case of spherical pure scattering in /30 the atmosphere, we can reduce the solution of the problem, in its exact formu-

lation, to the following equations of transfer of radiant energy:

$$\left. \begin{aligned} \cos \theta \frac{\partial I_1}{\partial \tau} &= K - I_1, \\ -\cos \theta \frac{\partial \tilde{I}_2}{\partial \tau} &= K - \tilde{I}_2, \end{aligned} \right\} \quad (1)$$

where I_1 and \tilde{I}_2 are the intensities of the rays directed away from the earth's surface and toward it, respectively; θ is the angle made with the vertical by the former of these rays; τ is the optical thickness of the layer of atmosphere from the earth's surface to a specified height z , and

$$K(\tau) = \frac{1}{2} \int_0^{\frac{\pi}{2}} I_1(\tau, \theta) \sin \theta d\theta + \frac{1}{2} \int_0^{\frac{\pi}{2}} \tilde{I}_2(\tau, \theta) \sin \theta d\theta. \quad (2)$$

Let $\tilde{I}_2(\tau^*, \theta)$ be the intensity of the light incident on the upper boundary of the atmosphere (for $\tau = \tau^*$). It is convenient to take, as a new unknown function, instead of $\tilde{I}_2(\tau, \theta)$, the function $I_2(\tau, \theta)$, defined by the transformation

$$\tilde{I}_2(\tau, \theta) = I_2(\tau, \theta) + e^{-(\tau^*-\tau) \sec \theta} \tilde{I}_2(\tau^*, \theta). \quad (3)$$

Obviously, for $\tau = \tau^*$, the following condition must hold:

$$I_2(\tau^*, \theta) = 0. \quad (4)$$

Then the systems of equations (1) and (2) take the form:

$$\left. \begin{aligned} \cos \theta \frac{\partial I_1}{\partial \tau} &= K - I_1, \\ -\cos \theta \frac{\partial I_2}{\partial \tau} &= K - I_2, \end{aligned} \right\} \quad (5)$$

$$\begin{aligned} K(\tau) &= \frac{1}{2} \int_0^{\frac{\pi}{2}} I_1(\tau, \theta) \sin \theta d\theta + \frac{1}{2} \int_0^{\frac{\pi}{2}} I_2(\tau, \theta) \sin \theta d\theta + \\ &+ \int_0^{\frac{\pi}{2}} e^{-(\tau^*-\tau) \sec \theta} \tilde{I}_2(\tau^*, \theta) \sin \theta d\theta. \end{aligned} \quad (6)$$

Below, we are concerned with the case in which the light is incident on the upper boundary of the atmosphere in the form of a parallel beam making the angle ζ (zenith distance of the sun) with the vertical. In this case, denoting by πS the flux across unit area normal to the beam, eq.(6) will be replaced /31

by

$$K(\tau) = \frac{1}{2} \int_0^{\frac{\pi}{2}} I_1(\tau, \theta) \sin \theta d\theta + \frac{1}{2} \int_0^{\frac{\pi}{2}} I_2(\tau, \theta) \sin \theta d\theta + \frac{S}{4} e^{-(\tau^* - \tau) \sec \zeta}. \quad (7)$$

The problem is thus reduced to the solution of three equations [eqs.(5) and (7)] in the unknown functions I_1 , I_2 , and K . The boundary condition for the function I_2 , at $\tau = \tau^*$, is expressed by eq.(4). To formulate the boundary conditions on the earth's surface ($\tau = 0$), we introduce the fluxes

$$\left. \begin{aligned} F_1(\tau) &= 2\pi \int_0^{\frac{\pi}{2}} I_1(\tau, \theta) \sin \theta \cos \theta d\theta, \\ F_2(\tau) &= 2\pi \int_0^{\frac{\pi}{2}} I_2(\tau, \theta) \sin \theta \cos \theta d\theta. \end{aligned} \right\} \quad (8)$$

Then, at $\tau = 0$, we have the condition

$$F_1(0) = q [F_2(0) + \pi S e^{-\tau^* \sec \zeta} \cos \zeta], \quad (9)$$

where q is the albedo of the earth's surface. In particular, at $q = 0$, we obtain

$$F_1(0) = 0. \quad (10)$$

1. The Schwarzschild Approximation

We now introduce the quantities

$$\left. \begin{aligned} K_1(\tau) &= 2\pi \int_0^{\frac{\pi}{2}} I_1(\tau, \theta) \sin \theta d\theta, \\ K_2(\tau) &= 2\pi \int_0^{\frac{\pi}{2}} I_2(\tau, \theta) \sin \theta d\theta. \end{aligned} \right\} \quad (11)$$

The Schwarzschild approximation is defined by a hypothesis according to which

$$\left. \begin{aligned} K_1(\tau) &\approx 2F_1(\tau), \\ K_2(\tau) &\approx 2F_2(\tau). \end{aligned} \right\} \quad (12)$$

Let us multiply eq.(5) by $2\pi \sin \theta$ and integrate over θ from 0 to $\frac{\pi}{2}$. Then,

$$\left. \begin{aligned} \frac{dF_1}{d\tau} &= 2\pi K - K_1, \\ -\frac{dF_2}{d\tau} &= 2\pi K - K_2, \end{aligned} \right\} \quad (13)$$

whence, making use of the conditions (12), we obtain the approximate equations:

$$\left. \begin{aligned} \frac{dF_1}{d\tau} &= 2\pi K - 2F_1, \\ -\frac{dF_2}{d\tau} &= 2F_2 - 2\pi K. \end{aligned} \right\} \quad (14)$$

Equation (7) may be rewritten in the form of

$$\pi K(\tau) = \frac{K_1(\tau) + K_2(\tau)}{4} + \frac{\pi S}{4} e^{-(\tau^*-\tau)\sec \zeta} \quad (15)$$

or, approximately,

$$\pi K(\tau) = \frac{F_1(\tau) + F_2(\tau)}{2} + \frac{\pi S}{4} e^{-(\tau^*-\tau)\sec \zeta}. \quad (16)$$

Thus, in the Schwarzschild approximation, the problem is reduced to the solution of the following equations:

$$\left. \begin{aligned} \frac{dF_1}{d\tau} &= 2\pi K - 2F_1, \\ -\frac{dF_2}{d\tau} &= 2F_2 - 2\pi K, \\ \pi K &= \frac{F_1 + F_2}{2} + \frac{\pi S}{4} e^{-(\tau^*-\tau)\sec \zeta} \end{aligned} \right\} \quad (A)$$

under the boundary conditions

$$\left. \begin{aligned} F_2(\tau^*) &= 0; \\ F_1(0) &= q [F_2(0) + \pi S \cos \zeta e^{-\tau^* \sec \zeta}]. \end{aligned} \right\} \quad (A')$$

The solution of this system gives no trouble and, for the function $K(\tau)$, leads to the formula

$$\pi K(\tau) = \pi S \left(\frac{1}{4} - \cos^2 \zeta \right) e^{-(\tau^*-\tau)\sec \zeta} + \frac{1}{2} c_2 - c_1 \tau, \quad (17)$$

where

$$\begin{aligned} c_1 &= \frac{-2\pi S(1-q) \cos \zeta \left[\left(\frac{1}{2} + \cos \zeta \right) + \left(\frac{1}{2} - \cos \zeta \right) e^{-\tau^* \sec \zeta} \right]}{(1-q)(2\tau^*+1)+1+q}; \\ c_2 &= \frac{2\pi S \cos \zeta \left[(1-q) \left(\frac{1}{2} + \cos \zeta \right) - (1-q)(2\tau^*+1) \left(\frac{1}{2} - \cos \zeta \right) e^{-\tau^* \sec \zeta} \right]}{(1-q)(2\tau^*+1)+1+q}. \end{aligned}$$

In particular, at $q = 0$, we have

$$c_1 = -\frac{\pi S \cos \zeta \left[\left(\frac{1}{2} + \cos \zeta \right) + \left(\frac{1}{2} - \cos \zeta \right) e^{-\tau^* \sec \zeta} \right]}{\tau^* + 1};$$

$$c_2 = -\frac{\pi S \cos \zeta \left[\left(\frac{1}{2} + \cos \zeta \right) - (2\tau^* + 1) \left(\frac{1}{2} - \cos \zeta \right) e^{-\tau^* \sec \zeta} \right]}{\tau^* + 1}.$$

In this case,

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$$\begin{aligned} \pi K(\tau) &= \pi S \left(\frac{1}{2} + \cos \zeta \right) \left[\cos \zeta + \left(\frac{1}{2} - \cos \zeta \right) e^{-(\tau^* - \tau) \sec \zeta} \right] - \\ &- \pi S \cos \zeta \left[\left(\frac{1}{2} + \cos \zeta \right) + \left(\frac{1}{2} - \cos \zeta \right) e^{-\tau^* \sec \zeta} \right] \frac{\tau^* - \tau + \frac{1}{2}}{\tau^* + 1}. \end{aligned} \quad (18)$$

2. The Eddington Approximation

We now introduce the quantities

$$\left. \begin{aligned} H_1(\tau) &= 2\pi \int_0^{\frac{\pi}{2}} I_1(\tau, 0) \cos^2 0 \sin 0 d\theta, \\ H_2(\tau) &= 2\pi \int_0^{\frac{\pi}{2}} I_2(\tau, 0) \cos^2 0 \sin 0 d\theta \end{aligned} \right\} \quad (19)$$

The Eddington approximation is defined by the hypothesis

$$\left. \begin{aligned} K_1(\tau) &\approx 3H_1(\tau), \\ K_2(\tau) &\approx 3H_2(\tau). \end{aligned} \right\} \quad (20)$$

Hence,

$$K_1(\tau) + K_2(\tau) \approx 3[H_1(\tau) + H_2(\tau)]$$

and

$$\pi K(\tau) = \frac{3}{4} [H_1(\tau) + H_2(\tau)] + \frac{\pi S}{4} e^{-(\tau^* - \tau) \sec \zeta}. \quad (21)$$

Adding and subtracting eq.(5), we obtain

$$\cos \theta \frac{\partial (I_1 + I_2)}{\partial \tau} = -(I_1 - I_2), \quad (22)$$

$$\cos \theta \frac{\partial (I_1 - I_2)}{\partial \tau} = 2K - (I_1 + I_2). \quad (23)$$

Let us multiply eq.(23) by $2\pi \sin \theta$ and integrate over θ from 0 to $\frac{\pi}{2}$. Then,

$$\frac{d(F_1 - F_2)}{d\tau} = 4\pi K - [K_1(\tau) + K_2(\tau)]$$

or

$$\frac{d(F_1 - F_2)}{d\tau} = \pi S e^{-(\tau^* - \tau) \sec \zeta},$$

whence

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$$F_1 - F_2 = \pi S \cos \zeta e^{-(\tau^* - \tau) \sec \zeta} + c_1. \quad (24)$$

Multiplying eq.(22) by $2\pi \cos \theta \sin \theta$ and integrating from 0 to $\frac{\pi}{2}$ will yield

$$\frac{d(H_1 + H_2)}{d\tau} = -(F_1 - F_2) \quad (25)$$

or, using eq.(24),

$$\frac{d(H_1 + H_2)}{d\tau} = -\pi S \cos \zeta e^{-(\tau^* - \tau) \sec \zeta} - c_1,$$

whence

$$H_1 + H_2 = c_2 - c_1 \tau - \pi S \cos^2 \zeta e^{-(\tau^* - \tau) \sec \zeta}.$$

Consequently,

$$\pi K(\tau) = \frac{\pi S}{4} e^{-(\tau^* - \tau) \sec \zeta} + \frac{3}{4} (c_2 - c_1 \tau - \pi S \cos^2 \zeta e^{-(\tau^* - \tau) \sec \zeta}). \quad (26)$$

Let us now calculate the constants c_1 and c_2 . Combining conditions (12) and (20), we may write:

$$\left. \begin{array}{l} H_1(\tau) \approx \frac{2}{3} F_1(\tau), \\ H_2(\tau) \approx \frac{2}{3} F_2(\tau). \end{array} \right\} \quad (27)$$

Consequently,

$$F_1 + F_2 \approx \frac{3}{2} [c_2 - c_1 \tau - \pi S \cos^2 \zeta e^{-(\tau^* - \tau) \sec \zeta}]. \quad (28)$$

From eqs.(24) and (28) we obtain, at $\tau = 0$ and $\tau = \tau^*$:

$$\left. \begin{array}{l} F_1(0) - F_2(0) = \pi S \cos \zeta e^{-\tau^* \sec \zeta} + c_1; \\ F_1(0) + F_2(0) = \frac{3}{2} [c_2 - \pi S \cos^2 \zeta e^{-\tau^* \sec \zeta}]; \\ F_1(\tau^*) - F_2(\tau^*) = \pi S \cos \zeta + c_1; \\ F_1(\tau^*) + F_2(\tau^*) = \frac{3}{2} [c_2 - c_1 \tau^* - \pi S \cos^2 \zeta]. \end{array} \right\} \quad (29)$$

However,

$$F_2(\tau^*) = 0$$

and

$$F_1(0) = q \cdot F_2(0),$$

whence

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$$\begin{aligned} F_1(0) - F_2(0) &= -(1-q) F_2(0), \\ F_1(0) + F_2(0) &= (1+q) F_2(0). \end{aligned}$$

Thus, for $F_1(\tau^*)$ we obtain the expressions:

$$F_1(\tau^*) = \pi S \cos \zeta + c_1$$

and

$$F_1(\tau^*) = \frac{3}{2} (c_2 - c_1 \tau^* - \pi S \cos^2 \zeta),$$

while, for $F_2(0)$,

$$F_2(0) = -\frac{\pi S \cos \zeta e^{-\tau^* \sec \zeta} + c_1}{1-q}$$

and

$$F_2(0) = -\frac{\frac{3}{2} (c_2 - \pi S \cos^2 \zeta e^{-\tau^* \sec \zeta})}{1+q}.$$

Hence, we obtain the following equations for determining c_1 and c_2 :

$$\left. \begin{aligned} \pi S \cos \zeta + c_1 &= \frac{3}{2} (c_2 - c_1 \tau^* - \pi S \cos^2 \zeta) \\ \frac{\pi S \cos \zeta e^{-\tau^* \sec \zeta} + c_1}{1-q} &= \frac{\frac{3}{2} (c_2 - \pi S \cos^2 \zeta e^{-\tau^* \sec \zeta})}{1+q} \end{aligned} \right\} \quad (30)$$

Hence,

$$c_1 = -\frac{\pi S \cos \zeta \left[\frac{4}{3} e^{-\tau^* \sec \zeta} + (1-q) \left(\cos \zeta + \frac{2}{3} \right) (1 - e^{-\tau^* \sec \zeta}) \right]}{\tau^*(1-q) + \frac{4}{3}}, \quad (31)$$

$$\begin{aligned} c_2 &= \pi S \cos \zeta \left(\cos \zeta + \frac{2}{3} \right) - \\ &- \left(\tau^* + \frac{2}{3} \right) \frac{\pi S \cos \zeta \left[\frac{4}{3} e^{-\tau^* \sec \zeta} + (1-q) \left(\cos \zeta + \frac{2}{3} \right) (1 - e^{-\tau^* \sec \zeta}) \right]}{\tau^*(1-q) + \frac{4}{3}}. \end{aligned}$$

At $q = 0$, we have

$$c_1 = -\frac{\pi S \cos \zeta \left[\frac{4}{3} e^{-\tau^* \sec \zeta} + \left(\cos \zeta + \frac{2}{3} \right) (1 - e^{-\tau^* \sec \zeta}) \right]}{\tau^* + \frac{4}{3}}, \quad (31)$$

$$c_2 = \pi S \cos \zeta \left(\cos \zeta + \frac{2}{3} \right) - \left. \begin{aligned} & \pi S \cos \zeta \left[\frac{4}{3} e^{-\tau^* \sec \zeta} + \left(\cos \zeta + \frac{2}{3} \right) (1 - e^{-\tau^* \sec \zeta}) \right] \\ & - \left(\tau^* + \frac{2}{3} \right) \frac{\tau^* \sec \zeta}{\tau^* + \frac{4}{3}} \end{aligned} \right\} \quad (32)$$

Substituting the values obtained for c_1 and c_2 into eq.(26), we find /36
for $K(\tau)$:

$$K(\tau) = \frac{S}{4} (1 - 3 \cos^2 \zeta) e^{-(\tau^* - \tau) \sec \zeta} + \frac{3}{4} S \cos \zeta \left[\cos \zeta + \frac{2}{3} - \left(\tau^* - \tau + \frac{2}{3} \right) \frac{\frac{4}{3} e^{-\tau^* \sec \zeta} + (1-q) \left(\cos \zeta + \frac{2}{3} \right) (1 - e^{-\tau^* \sec \zeta})}{\tau^* (1-q) + \frac{4}{3}} \right]. \quad (33)$$

In particular, at $q = 0$,

$$K(\tau) = \frac{S}{4} (1 - 3 \cos^2 \zeta) e^{-(\tau^* - \tau) \sec \zeta} + \frac{3}{4} S \cos \zeta \left[\cos \zeta + \frac{2}{3} - \left(\tau^* - \tau + \frac{2}{3} \right) \frac{\frac{4}{3} e^{-\tau^* \sec \zeta} + \left(\cos \zeta + \frac{2}{3} \right) (1 - e^{-\tau^* \sec \zeta})}{\tau^* + \frac{4}{3}} \right]. \quad (34)$$

Next, the approximate values of $K(\tau)$ were compared for the case of $q = 0$ with the exact values obtained by numerical solution of the integral equation of light scattering. The following values of the parameters τ^* and ζ were used in this comparison:

$$\begin{aligned} \tau^* &= 0.2; 0.3; 0.4; 0.5; 0.6; \\ \zeta &= 30, 45, 60, 76^\circ. \end{aligned}$$

For each of the 20 combinations of these two parameters we calculated $K(\tau)$ by means of the formulas:

$$\frac{1}{S} K_1(\tau) = \frac{1}{4} e^{-(\tau^* - \tau) \sec \zeta}; \quad (35)$$

$$\begin{aligned} \frac{1}{S} K_{sch}(\tau) &= \left(\frac{1}{2} + \cos \zeta \right) \left[\cos \zeta + \left(\frac{1}{2} - \cos \zeta \right) e^{-(\tau^* - \tau) \sec \zeta} \right] - \\ &- \cos \zeta \left[\left(\frac{1}{2} + \cos \zeta \right) + \left(\frac{1}{2} - \cos \zeta \right) e^{-\tau^* \sec \zeta} \right] \frac{\tau^* - \tau + \frac{1}{2}}{\tau^* + \frac{1}{2}}; \end{aligned} \quad (36)$$

$$\frac{1}{S} K_e(\tau) = \frac{3}{4} \left[\cos \zeta \left(\cos \zeta + \frac{2}{3} \right) + \left(\frac{1}{3} - \cos^2 \zeta \right) e^{-(\tau^* - \tau) \sec \zeta} \right] -$$

$$-\frac{3}{4} \cos \zeta \left[\frac{2}{3} + \cos \zeta + \left(\frac{2}{3} - \cos \zeta \right) e^{-\tau^* \sec \zeta} \right] \frac{\tau^* - \tau + \frac{2}{3}}{\tau^* + \frac{4}{3}}. \quad (37)$$

We used the values of $\varphi_1 = \frac{K_1(\tau)}{S}$, $\varphi_{sch} = \frac{K_{sch}(\tau)}{S}$, and $\varphi_e(\tau) = \frac{K_e(\tau)}{S}$ calculated from eqs.(35), (36), and (37) for comparing with the exact solution of the integral equation $\varphi(\tau)$.

The results of the comparison may be summarized in the following propositions: /37

- 1) All three approximate solutions $\varphi_1(\tau)$, $\varphi_{sch}(\tau)$, and $\varphi_e(\tau)$, are lower than the exact solution $\varphi(\tau)$ of the integral equation.
- 2) The Schwarzschild approximation $\varphi_{sch}(\tau)$ is somewhat closer to the exact solution $\varphi(\tau)$ than is the Eddington solution $\varphi_e(\tau)$; however, the difference between these two approximations is very small, and in most practical cases they may be considered as coinciding.
- 3) The Schwarzschild approximation $\varphi_{sch}(\tau)$ gives a result closer to the exact solution than does the approximation $\varphi_1(\tau)$, which allows only for single scattering. It may roughly be considered that the Schwarzschild approximation is equivalent to allowance for double or plural scattering.

A more exact idea of the relations between the various approximations and the exact solution can be obtained from the Tables.

V. ON THE EXACTNESS OF CHANDRASEKHAR'S APPROXIMATE METHOD

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B.V.Ovchinskiy

The object of this paper is to compare the solution of the equation of transfer of radiant energy, based on the method recently proposed by Chandrasekhar, with the solution obtained from the integral equation (1) of the first paper in this collection, based on the method of successive approximations.

It is well known that the equation of radiant energy transfer for a spherical scattering indicatrix may be represented in the form of

$$\cos \theta \frac{\partial I}{\partial \tau} = \frac{1}{2} \int I \sin \theta d\theta - I(\tau, \theta). \quad (1)$$

To make use of the method proposed by Chandrasekhar, let us put $\tau = \tau^* - \tau'$ and $\cos \theta = \mu$. Then eq.(1) may be replaced by

$$\mu \frac{dI}{d\tau'} = I - \frac{1}{2} \int_{-1}^{+1} I d\mu. \quad (2)$$

The principle of the approximate method developed by Chandrasekhar for the solution of this equation consists in substituting the integral by the sum according to Gauss' method. Equation (2) can then be replaced by the following system of differential equations:

$$\mu_i \frac{dI_i}{d\tau'} = I_i - \frac{1}{2} \sum_{j=-n}^{n} a_j I_j \quad (i = \pm 1, \pm 2, \dots, \pm n), \quad (3)$$

where the quantities a_j are the Gaussian coefficients, and μ_i the roots of a Legendre polynomial of the $2n$ order. The solution of this system is found, as always, in the form of $g_i e^{-k_i \tau'}$; k is found as a root of the equation:

$$I = \sum_{i=1}^n \frac{a_i}{1 - \mu_i^2 k^2}.$$

The solution of the system (3) may be written in the form of

$$I_i = b \left\{ \sum_{\alpha=1}^{n-1} \left[\frac{L_\alpha e^{-k_\alpha \tau'}}{1 + \mu_i k_\alpha} + \frac{L_{-\alpha} e^{+k_\alpha \tau'}}{1 - \mu_i k_\alpha} \right] + \mu_i + Q + \tau' \right\}$$

or, passing to the old independent variable,

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$$I_i = b \left\{ \sum_{\alpha=1}^{n-1} \left[\frac{L_\alpha e^{-k_\alpha(\tau^*-\tau)}}{1+\mu_i k_\alpha} + \frac{L_{-\alpha} e^{k_\alpha(\tau^*-\tau)}}{1-\mu_i k_\alpha} \right] + \mu_i + Q + \tau^* - \tau \right\} \quad (4)$$

$$(i = \pm 1, \pm 2, \dots, \pm n).$$

To determine the arbitrary constants L_α , $L_{-\alpha}$, b , and Q , use must be made of boundary conditions which, in the case of diffuse radiation of the earth, can be written in the following form:

$$I_{-i} = I^{(2)}(\tau^*) = 0, \quad I_i = I^{(1)}(0) = I_s \quad (i = \pm 1, \pm 2, \dots, \pm n). \quad (5)$$

In the first approximation ($n = 1$), we obtain

$$I_1 = \frac{\sqrt{3} I_s}{2 + \sqrt{3} \tau^*} \left[\frac{2}{\sqrt{3}} + \tau^* - \tau \right], \quad I_{-1} = \frac{\sqrt{3} I_s}{2 + \sqrt{3} \tau^*} (\tau^* - \tau). \quad (6)$$

Having the quantities I_1 , I_{-1} , we easily establish that

$$K = \frac{1}{2} \int I \sin \theta d\theta \approx I_s \frac{1 + \sqrt{3}(\tau^* - \tau)}{2 + \sqrt{3}\tau^*}. \quad (7)$$

The second approximation is obtained by putting $n = 2$ (for $i = 1, 2$):

$$\left. \begin{aligned} I_1 &= b \left[\frac{L_1 e^{-k(\tau^*-\tau)}}{1+\mu_1 k} + \frac{L_{-1} e^{k(\tau^*-\tau)}}{1-\mu_1 k} + \mu_1 + Q + \tau^* - \tau \right], \\ I_2 &= b \left[\frac{L_1 e^{-k(\tau^*-\tau)}}{1-\mu_2 k} + \frac{L_{-1} e^{k(\tau^*-\tau)}}{1+\mu_2 k} - \mu_2 + Q + \tau^* - \tau \right], \end{aligned} \right\} \quad (8)$$

$$k = 1.97203, \quad \mu_1 = 0.399981, \quad \mu_2 = 0.861136.$$

The constants L_1 , L_{-1} , b , and Q are here determined from the system of equations derived from eqs.(8) at $\tau = 0$ and $\tau = \tau^*$, if the boundary conditions (5) are borne in mind.

For K we obtain the approximate expression

$$K \approx b [\tau^* - \tau + Q + L_1 e^{-k(\tau^*-\tau)} + L_{-1} e^{k(\tau^*-\tau)}]. \quad (9)$$

Table (XI.1) gives the values of the quantity $\omega(\tau) = \frac{K(\tau)}{I_s}$ calculated from eqs.(7) and (9) and denoted by Ch_1 and Ch_2 . The same Table also gives the values of $\omega(\tau)$ obtained by solution of the integral equation:

$$\omega(\tau) = \frac{1}{2} E_2(\tau) + \frac{1}{2} \int_0^{\tau^*} \omega(t) E_1(|\tau - t|) dt$$

using the method of successive approximations. The same Table also gives the 140 values of $\omega(\tau)$ calculated by means of Schwarzschild's method

$$\omega(\tau) \approx \frac{\tau^* - \tau + \frac{1}{2}}{\tau^* + 1}.$$

It is obvious from the calculation results that the values of $\omega(\tau)$ obtained by the Schwarzschild method are of an exactness intermediate between the first and second Chandrasekhar approximations. Several points in the case of $\tau^* = 0.6$ are an exception.

Let us pass to the case when the atmosphere is illuminated from above by a parallel beam of rays of intensity πS making the angle ζ with the vertical. In this case, the boundary conditions (5) may be retained, if the radiant energy transfer equation is taken in the form of

$$\cos \theta \frac{\partial I}{\partial \tau} = \frac{1}{2} \int I \sin \theta d\theta + \frac{1}{4} S e^{-(\tau^* - \tau) \sec \zeta} - I. \quad (10)$$

Putting $\cos \theta = \mu$ and $\tau = \tau^* - \tau'$, eq.(10) may be represented as follows:

$$\mu \frac{\partial I}{\partial \tau'} = I - \frac{1}{2} \int_{-1}^{+1} I d\mu - \frac{S}{4} e^{-\tau' \sec \zeta}$$

or, using Chandrasekhar's method,

$$\mu_i \frac{dI_i}{d\tau'} = I_i - \frac{1}{2} \sum_{j=-n}^{j=n} a_j I_j - \frac{S}{4} e^{-\tau' \sec \zeta}. \quad (11)$$

Thus, we obtain a system of $2n$ inhomogeneous differential equations. The solution of this system is the sum of the general solution of the homogeneous system and the particular solution of the inhomogeneous system. The solution of the homogeneous system is of the form of eqs.(4).

Let us seek the particular solution in the form of $H_i e^{-\tau' \sec \zeta}$. Substituting this particular solution into eq.(9), we have

$$H_i (1 + \mu_i \sec \zeta) = \frac{S}{4} + \frac{1}{2} \sum_{j=-n}^{j=n} a_j H_j = B \quad (12)$$

or

$$H_t = \frac{B}{1 + \mu_t \sec \zeta}.$$

Substituting H_i of eq.(12) by the resultant expression, we obtain the following value for B:

$$B = \frac{S}{4 \left(1 - \sum_{i=1}^n \frac{a_i}{1 - \mu_i^2 \sec^2 \zeta} \right)},$$

so that the particular solution of the system may be represented in the form of /41

$$\frac{Se^{-\tau' \sec \zeta}}{4(1 + \mu_i \sec \zeta) \left(1 - \sum_{i=1}^n \frac{a_i}{1 - \mu_i^2 \sec^2 \zeta} \right)}. \quad (13)$$

Passing to variable τ , we represent the general solution as follows:

$$I_i = b \left\{ \sum_{\alpha=1}^{n-1} \left[\frac{L_\alpha e^{-k_\alpha(\tau^*-\tau)}}{1 + \mu_i k_\alpha} + \frac{L_{-\alpha} e^{k_\alpha(\tau^*-\tau)}}{1 - \mu_i k_\alpha} \right] + \mu_i + Q + \tau^* - \tau \right\} + \\ + \frac{Se^{-(\tau^*-\tau) \sec \zeta}}{4(1 + \mu_i \sec \zeta) \left(1 - \sum_{i=1}^n \frac{a_i}{1 - \mu_i^2 \sec^2 \zeta} \right)}; \quad (14)$$

for the case of $k_\alpha \neq \sec \zeta$. In the case of $k_\alpha = \sec \zeta$, the particular solution must be sought in a different form.

The constants L_α , $L_{-\alpha}$, b , and Q are determined from the boundary conditions (5) or, as we will do below, from

$$I_i(0) = 0, \quad I_{-i}(\tau^*) = 0 \quad (i = 1, 2, 3, \dots, n). \quad (15)$$

For further calculation, let us now write out the formulas of the first and second approximations.

First Approximation ($n = 1$)

The solution of the system (11) is of the form

$$I_1 = b \left(\frac{1}{V3} + Q + \tau^* - \tau \right) + \frac{\sqrt{3}}{4} S \frac{\sec \zeta - \sqrt{3}}{\sec^2 \zeta} e^{-(\tau^*-\tau) \sec \zeta}, \\ I_{-1} = b \left(-\frac{1}{V3} + Q + \tau^* - \tau \right) - \frac{\sqrt{3}}{4} S \frac{\sec \zeta + \sqrt{3}}{\sec^2 \zeta} e^{-(\tau^*-\tau) \sec \zeta}. \quad \left. \right\} \quad (16)$$

Hence,

$$K(\tau) \approx \frac{I_1 + I_{-1}}{2} + \frac{S}{4} e^{-(\tau^* - \tau) \sec \zeta} = \\ = b(Q + \tau^* - \tau) + \frac{S}{4}(1 - 3 \cos^2 \zeta) e^{-(\tau^* - \tau) \sec \zeta}. \quad (17)$$

Using the boundary conditions (15), we obtain from eqs.(16) the values of b and Q :

$$b = -\frac{3}{4} S \cos \zeta \left. \begin{array}{l} \frac{\cos \zeta + \frac{1}{V^3} - (\cos \zeta - \frac{1}{V^3}) e^{-\tau^* \sec \zeta}}{\tau^* + \frac{2}{V^3}} \\ \frac{(\frac{1}{V^3} + \tau^*) (\cos \zeta + \frac{1}{V^3}) + \frac{1}{V^3} (\cos \zeta - \frac{1}{V^3}) e^{-\tau^* \sec \zeta}}{\tau^* + \frac{2}{V^3}} \end{array} \right\}; \\ Q = \frac{3}{4} S \cos \zeta \left. \begin{array}{l} \frac{(\frac{1}{V^3} + \tau^*) (\cos \zeta + \frac{1}{V^3}) + \frac{1}{V^3} (\cos \zeta - \frac{1}{V^3}) e^{-\tau^* \sec \zeta}}{\tau^* + \frac{2}{V^3}} \\ \frac{\cos \zeta + \frac{1}{V^3} - (\cos \zeta - \frac{1}{V^3}) e^{-\tau^* \sec \zeta}}{\tau^* + \frac{2}{V^3}} \end{array} \right\} \quad (18)$$

Substituting these values of b and bQ into eq.(17), we reduce the expression 142 for $K(\tau)$, after simple computations, to the form:

$$K(\tau) \approx \frac{3}{4} S \left(\cos \zeta + \frac{1}{V^3} \right) \left[\cos \zeta - \left(\cos \zeta - \frac{1}{V^3} \right) e^{-(\tau^* - \tau) \sec \zeta} \right] - \\ - \frac{3}{4} S \cos \zeta \left[\cos \zeta + \frac{1}{V^3} - \left(\cos \zeta - \frac{1}{V^3} \right) e^{-\tau^* \sec \zeta} \right] \frac{\frac{\tau^* - \tau + \frac{1}{V^3}}{\tau^* + \frac{2}{V^3}}}{}, \quad (19)$$

indicating complete analogy to the approximations of Schwarzschild (IV.18) and Eddington (IV.37).

The results of calculations by eq.(19) for several values of τ^* and ζ are given in Table XI.2, Column Ch₁.

Second Approximation ($n = 2$)

In this case the solution is of the form (putting $S = 1$)

$$I_i = b \left[\frac{L_1 e^{-k(\tau^* - \tau)}}{1 + \mu_i k} + \frac{L_{-1} e^{k(\tau^* - \tau)}}{1 - \mu_i k} + \mu_i + Q + \tau^* - \tau \right] + \\ + \frac{e^{-(\tau^* - \tau) \sec \zeta} (1 - \mu_1^2 \sec^2 \zeta) (1 - \mu_2^2 \sec^2 \zeta)}{4 \left(\mu_1^2 \mu_2^2 \sec^2 \zeta - \frac{1}{3} \right) \sec^2 \zeta (1 + \mu_i \sec \zeta)} \quad (i = \pm 1, \pm 2); \quad (20)$$

$$\varphi(\tau) = b [L_1 e^{-k(\tau^* - \tau)} + L_{-1} e^{k(\tau^* - \tau)} + Q + \tau^* - \tau] +$$

$$+ \frac{1 - (\mu_2^2 + \mu_1^2 - \frac{1}{3}) \sec^2 \zeta}{4(\mu_1^2 \mu_2^2 \sec^2 \zeta - \frac{1}{3}) \sec^2 \zeta} e^{-(\tau^* - \tau) \sec \zeta}; \quad (21)$$

$$k = 1.97203, \mu_1 = 0.33998, \mu_2 = 0.86114.$$

To determine the constants, eq.(20) and the boundary conditions (15) must be used, yielding a system of four equations in four unknowns. Solving this system, we obtain the following expressions:

$$\left. \begin{aligned} L_{-1} &= \frac{\mu_1 [t(1+q) - s(1-p)] + \mu_2 [s(1+p) - t(1+q)] + st^* p}{4.4643 e^{k^*} (s-t) + 0.22803 qt + sp (1.43538 e^{k^*} + 0.59886)}; \\ L_1 &= \frac{0.52116 (1+q) - (4.4643 e^{k^*} - 0.22803 q) L_{-1}}{s}; \\ b &= \frac{m-n}{4.4643 L_1 + 0.22803 L_{-1} + 0.52116}; \\ Q &= \frac{n}{3.0289} + 1.43538 L_1 - 0.37083 L_{-1} + 0.86114, \end{aligned} \right\} \quad (22)$$

where, for brevity, the following notation is introduced: 143

$$\begin{aligned} m &= \frac{(1 + \mu_1 \sec \zeta)(1 - \mu_2^2 \sec^2 \zeta)}{4(\mu_1^2 \mu_2^2 \sec^2 \zeta - \frac{1}{3}) \sec^2 \zeta}; \quad n = \frac{(1 - \mu_1^2 \sec^2 \zeta)(1 + \mu_2 \sec \zeta)}{4(\mu_1^2 \mu_2^2 \sec^2 \zeta - \frac{1}{3}) \sec^2 \zeta}; \\ p &= \frac{\sec \zeta e^{-\tau^* \sec \zeta} (1 - \mu_1 \sec \zeta) \cdot 0.52116}{(1 + \mu_1 \sec \zeta) [e^{-\tau^* \sec \zeta} (1 - \mu_1 \sec \zeta) - (1 + \mu_2 \sec \zeta)]}; \\ q &= \frac{e^{-\tau^* \sec \zeta} (1 - \mu_1 \sec \zeta) (1 - \mu_2 \sec \zeta)}{(1 + \mu_1 \sec \zeta) (1 + \mu_2 \sec \zeta)}; \\ s &= 0.22803 e^{-k^*} - 4.4643 q; \\ t &= 0.22803 e^{-k^*} - p (0.37083 e^{-k^*} - 3.0289). \end{aligned}$$

It is more convenient to begin the calculation of the constants L, b, and Q by calculating the auxiliary quantities m, n, p, q, s, t and then to pass to eq.(22). As will be seen from the above formulas, the calculations become cumbersome already at the second approximation. The introduction of three or more approximations would require the solution of a system of six or more equations each time. Table XI.2, Column Ch₂, gives the values of $\varphi(\tau)$ according to the second approximation of Chandrasekhar. The same Table also gives the values of $\varphi(\tau)$ by the Schwarzschild method and the error (in percent) for Ch₁, Ch₂, and for the Schwarzschild approximation as compared with the exact solution (by successive approximations).

A study of this Table (for those values of τ^* and ζ for which calculations were performed) leads to the following conclusions:

The first approximation according to Chandrasekhar (Ch₁) gives less exact

results than that obtained by the Schwarzschild method.

The second approximation comes closer to the exact solution than the Schwarzschild approximation.

In conclusion, we note that the choice of the first approximation described by us in Paper I of this book gives results somewhat closer to the exact solution than the second Chandrasekhar approximation. This parabola method again does not require increased computational work.

The graphs at the end of this book give the result of calculations by the Chandrasekhar method (second approximation). A study of these graphs confirms our above statements.

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VI. APPLICATION OF THE NUMERICAL SOLUTION OF THE INTEGRAL
EQUATION OF THE THEORY OF LIGHT SCATTERING IN THE
ATMOSPHERE TO THE COMPILATION OF TABLES OF
THE HAZE FACTOR

Ye.S.Kuznetsov

A peculiar optical characteristic of the atmosphere, the so-called "haze factor", is very important in problems of aerial photogrammetry and in other applications of the theory of light scattering in the atmosphere. If B is the true brightness of an object on the earth's surface, which is observed in a certain direction, T the transmission coefficient of the atmosphere for the same direction, and β the corresponding brightness of the haze, then the haze factor will be defined by the formula

$$\sigma = \frac{\beta}{B \cdot T}. \quad (1)$$

The brightness B of the object is proportional to the illumination E of the earth's surface, the "brightness factor" r being the proportionality factor and depending in general on the direction of the incident ray and on that of the reflected ray. Thus we obtain the following formula for B :

$$B = rE, \quad (2)$$

and for the haze factor:

$$\sigma = \frac{\beta}{rET}. \quad (3)$$

In what follows, we shall consider the case when the brightness factor r is independent of both the direction of the incident ray and that of the reflected ray. In this case the reflectivity of the object is characterized by the albedo Q , which is connected with the brightness factor by the formula

$$Q = \pi r. \quad (4)$$

Then, for the haze factor we have

$$\sigma(Q) = \frac{\pi\beta}{QET}. \quad (5)$$

The case $Q = 1$ corresponds to an ideal white reflecting surface. In this case, we obtain what is known as the "absolute haze factor":

$$\sigma = \frac{\pi\beta}{ET}. \quad (6)$$

The haze factor, for arbitrary Q , may be expressed in terms of the absolute haze factor by means of the equality

$$\sigma(Q) = \frac{\sigma}{Q}. \quad (7)$$

The absolute haze factor is a quantity depending in a very complex manner on a number of variables and physical parameters. Even if the problem is idealized to a considerable extent, we still must consider σ as being a function of the altitude z above the earth's surface and of the angle θ included between the direction of the visual ray and the direction of the vertical, and also as depending on three parameters: the optical thickness of the entire atmosphere τ^* , the zenith distance of the sun ζ , and the albedo of the earth's surface q . Here we must differentiate between the albedo Q of the observed object and the albedo q of the earth's surface. In particular, in the case of the absolute haze factor, we are dealing with an object of rather small angular diameter having an albedo $Q = 1$ and, observed against the background of the earth's surface, having an albedo q . We will denote this absolute haze factor by the symbol σ_q . The haze factor for the albedo q of the earth's surface and the

albedo Q of the object will then be $\sigma^{(q)} = \frac{\sigma_q}{Q}$.

Earlier attempts to calculate the haze factor theoretically (Faas, Levin) were only made to the first approximation, neglecting the "multiplicity" of the scattering of light and the reflection of light from the earth's surface.

The solution we propose is based on the following assumptions:

- 1) The optical properties of the atmosphere and the earth's surface do not depend on the horizontal coordinates.
- 2) The atmosphere is a medium with pure scattering (with no true absorption of light).
- 3) The scattering of light at each point of the aerial medium proceeds by a spherical law (spherical scattering indicatrix).
- 4) The atmosphere is illuminated by a parallel beam of rays emitted by the sun; this beam encounters no obstacles, such as clouds, in its propagation through the atmosphere.

The exact solution of the problem of the scattering of light in the atmosphere under these conditions reduces to a solution of the integral equation (II.32).

In the notation of Paper II of this book, we then have 146

$$\beta = S \cdot \sec \theta \int_0^{\tau^*} e^{-(\tau-t) \sec \theta} \varphi_q(t) dt; \quad (8)$$

$$E = \pi S e^{-\tau^* \sec \zeta} \cos \zeta + 2 \pi S \int_0^{\tau^*} \varphi_q(t) E_2(t) dt; \quad (9)$$

$$T = e^{-\tau \sec \theta}. \quad (10)$$

Substituting eqs.(8) - (10) into eq.(6), we obtain the following expression for the absolute haze factor at an albedo q for the earth's surface:

$$\sigma_q = \frac{\sec \theta \int_0^\tau e^{t \sec \theta} \varphi_q(t) dt}{\cos \zeta e^{-\tau^* \sec \zeta} + 2 \int_0^{\tau^*} \varphi_q(t) E_2(t) dt}. \quad (11)$$

In particular, at $q = 0$, we obtain the formula:

$$\sigma_0 = \frac{\sec \theta \int_0^\tau e^{t \sec \theta} \varphi_0(t) dt}{\cos \zeta e^{-\tau^* \sec \zeta} + 2 \int_0^{\tau^*} \varphi_0(t) E_2(t) dt}, \quad (12)$$

where $\varphi_0(t)$ is the solution of the integral equation (1) in Paper II [eq.(II.1)]. The quantity σ_q can be expressed in terms of σ_0 , q and in terms of the solution $w(\tau)$ of the auxiliary integral equation of Paper II. For this, it is sufficient to replace the expression $\varphi_q(\tau)$ in eq.(11) by $\varphi_0(\tau)$ and $w(\tau)$ according to eq.(II.35) and make use of eq.(II.36). Simple computations will then yield the following important formula:

$$\sigma_q = \sigma_0 + q \left[\cos \theta \int_0^\tau e^{t \sec \theta} w(t) dt - 2 \sigma_0 \int_0^{\tau^*} w(t) E_2(t) dt \right], \quad (13)$$

showing that the absolute haze factor depends strictly linearly on the albedo of the earth's surface.

By the aid of eq.(13) the absolute haze factor for any albedo q of the earth's surface may be expressed in terms of σ_0 and q , and also in terms of the auxiliary integral equation (II.34).

Using eq.(12), on the basis of the numerical solution of the integral equation (II.1), we constructed Tables of the absolute haze factor σ_0 at an albedo $q = 0$ of the earth's surface.

To calculate the haze factor σ_q , use can be made either of eq.(13) or of 47 the formula

$$\sigma_q = \sigma_0 \left(1 - 2 q \int_0^{\tau^*} w(t) E_2(t) dt \right) + q \sec \theta \int_0^\tau e^{t \sec \theta} w(t) dt, \quad (14)$$

which is more convenient for computation, while special Tables may be compiled for the coefficients

$$1 - 2q \int_0^{\tau^*} \omega(t) E_2(t) dt \text{ and } q \sec \theta \int_0^{\tau} e^{t \sec \theta} \omega(t) dt$$

For calculations that do not require high accuracy, an approximate formula taking only single scattering into account may be derived from eq.(12). Putting, approximately,

$$\varphi_0(t) \approx \frac{1}{4} e^{-\tau^*(\tau^* - t) \sec \zeta}, \quad (15)$$

we will have:

$$\sigma_0 \approx \frac{1}{4} \frac{\cos \zeta}{\cos \zeta + \cos \theta} \frac{e^{\tau(\sec \zeta + \sec \theta)} - 1}{\cos \zeta + \frac{1}{2} \int_0^{\tau^*} e^{x \sec \zeta} E_2(x) dx}. \quad (16)$$

The haze factor is of great practical importance in aerial photography, but is also useful in other applications of the theory of light scattering in the atmosphere.

Table I; Functions $E_n(x)$ ($n = 1, 2, 3, 4$)

This Table gives the functions $E_1(x)$, $E_2(x)$, $E_3(x)$, and $E_4(x)$ to eight places, for values of the argument from $x = 0$ to $x = 0.6$ at intervals of 0.01. The coverage of the Table is intended for applications to problems of atmospheric optics, in which the highest value of x is determined by the optical thickness of the atmosphere, which rarely exceeds 0.6. The choice of the number of places and the interval between the values of the argument was dictated by the required accuracy of calculation in the numerical solution of the integral equation of light scattering.

This coverage of the Tables is insufficient for problems involving radiant heat exchange in the atmosphere and must be considerably extended for such uses.

Table II; Coefficients P_{kn} and Q_{kn}

In Paper II of this book, we gave the determination and method of application of the coefficients P_{kn} and Q_{kn} to the numerical solution of the integral equation of the theory of light scattering. The properties of the symbols P_{kn} and Q_{kn} make it possible to confine the presentation to the values of P_{0n} , Q_{0n} , and Q_{nn} . The same paper also gives a method for passing from these values of the symbols to their values for arbitrary values of k . The Table is presented in two versions, for the interval $h = 0.01$ and for $h = 0.02$, in view of the fact that the numerical solution of the integral equation had one value of h or the other, depending on the optical thickness of the atmosphere.

Table III; Solution of the Auxiliary Integral Equation $\omega(\tau)$

This Table contains the results of the numerical solution of the integral equation (II.34) at optical thicknesses of the atmosphere of $\tau^* = 0.2, 0.3, 0.4, 0.5$, and 0.6 . The solution was performed by the method of successive approximations, with eq.(II.39) used as the zero-th approximation. To reach the required accuracy, it was sufficient to calculate three approximations (not counting the zero-th) in each case.

The function $\omega(\tau)$ plays an intermediate role and is necessary for the transition from the solution of the integral equation of the theory of light scattering at an albedo of $q = 0$ of the earth's surface to the case of an arbitrary albedo.

Table IV: Solution of the Integral Equation of the Theory of Light Scattering, $\varphi_0(\tau)$

This Table consists of five subtables containing the results of the numeri-

cal solution of the fundamental integral equation (33) of the theory of light scattering in the atmosphere at the following values for the parameters τ^* (optical thickness of the atmosphere) and ζ (zenith distance of the sun):

$$\begin{aligned}\tau^* &= 0.2; 0.3; 0.4; 0.5; 0.6; \\ \zeta &= 30, 45, 60, 76^\circ.\end{aligned}$$

The albedo of the earth's surface is taken as zero in all 20 cases. The zero-th approximation was selected differently for different values of τ^* . For $\tau^* = 0.2$ and $\tau^* = 0.3$, eq.(II.38) which represents the Schwarzschild approximation (cf. Paper IV) was taken as the zero-th approximation, except for the case $\tau^* = 0.3$, $\zeta = 60^\circ$, in which the zero-th approximation adopted was the approximate solution of the integral equation in the paper by Ye.S.Kuznetsov (cf. Izv. AN SSSR, ser. geograf. i geofiz., Vol.9, p.204, 1945). For $\tau^* = 0.4, 0.5$, and 0.6 , the zero-th approximation was considered in the form of the equation

$$\tilde{\phi} + A + B\tau + C\tau^2,$$

where $\tilde{\phi}$ is the Schwarzschild approximation and A, B, and C are coefficients calculated by the method discussed in Paper III.

The number of successive approximations was in each case determined by the required accuracy. In the cases of $\tau^* = 0.2$ and $\tau^* = 0.3$, five approximations each were calculated (except for the case $\tau^* = 0.3$, $\zeta = 60^\circ$, in which only one approximation was calculated). For $\tau^* = 0.4$, three approximations proved sufficient and for $\tau^* = 0.5$ and $\tau^* = 0.6$, four approximations each (not counting the zero-th approximation in either case).

When Table IV is used to solve a specific problem of atmospheric optics, it must be borne in mind that the function $\phi_0(\tau)$ is the solution of the "reduced" integral equation of the theory of light scattering in the atmosphere, obtained under the assumption that the solar constant has a value equal to $\pi = 3.1415\dots$. If πS is the true value of the solar constant, then the quantity $S\phi_0(\tau)$, which is a solution of the unreduced equation, will have the physical meaning of the quantity of radiant energy scattered at an optical height τ by unit volume of the medium in unit solid angle (in any direction).

Formulas for other optical characteristics of the atmosphere in terms of the function $\phi_0(\tau)$ can be found in the specialized papers listed in the Introduction of the present book.

Table V; Coefficients H

Five Tables (V.1 - V.5) contain the values of the quantity H, calculated by eq.(II.36) or

$$H = \frac{q(\cos \zeta e^{-\tau^* \sec \zeta} + 2L)}{1 - 2qM}.$$

The values of L and M are given in Table VI, while the values of $\cos \zeta e^{-\tau^* \sec \zeta}$ are given in Table VIII. The coefficient H is used for passing from the solutions $\omega(\tau)$ and $\varphi_0(\tau)$ to the solution $\varphi_q(\tau)$ of the integral equation for an arbitrary albedo. /50

Table VI; Integrals L and M

Table VI contains the values of the integrals

$$L = \int_0^{\tau^*} \varphi_0(t) E_2(t) dt$$

and

$$M = \int_0^{\tau^*} \omega(t) E_2(t) dt$$

at various values of the parameters τ^* and ζ . The coefficient H (Table V) and certain other optical characteristics of the atmosphere are expressed in terms of these integrals.

Table VII; Solution of the Integral Equation of the Theory of Light Scattering

This Table, comprising 20 subtables corresponding to various combinations of τ^* and ζ , contains the solutions of the integral equation (II.32) at the following values of the albedo of the earth's surface:

$$q = 0.1; 0.2; 0.3; 0.8.$$

The first three of these values are for summer conditions, while the fourth is for winter (0.8 is the albedo of snow).

The calculations were performed by means of the formula

$$\varphi_q(\tau) = \varphi_0(\tau) + H \omega(\tau),$$

while the values of $\varphi_0(\tau)$, H, and $\omega(\tau)$ were taken, respectively, from Tables IV, V, and III. For convenience of comparison, the values of $\varphi_0(\tau)$ are repeated in the first column of the Tables. The solution $\varphi_q(\tau)$ for any value of q not covered by the Table is readily obtained by means of the above formula and Tables III - V.

Table VIII; Illumination of the Earth's Surface by Direct Solar Radiation

The quantity

$$\cos \zeta e^{-\tau^* \sec \zeta},$$

whose values at various values of ζ (zenith distance of the sun) and of τ^* (optical thickness of the atmosphere) are given in this Table, represents the luminous flux incident on the earth's surface when the upper boundary of the /51 atmosphere is illuminated by a parallel beam making an angle ζ with the vertical and having a luminous intensity of unity. If πS is the value of the solar constant, then the luminous flux incident on the earth's surface will have the value:

$$\pi S \cos \zeta e^{-\tau^* \sec \zeta}.$$

Table IX; Total Illumination of the Earth's Surface

This Table gives the values of the quantity

$$\begin{aligned} \cos \zeta e^{-\tau^* \sec \zeta} + 2 \int_0^{\tau^*} \varphi_q(t) E_2(t) dt &= \cos \zeta e^{-\tau^* \sec \zeta} + 2(L + HM) = \\ &= \frac{\cos \zeta e^{-\tau^* \sec \zeta} + 2L}{1 - 2qM} = \frac{H}{q}, \end{aligned}$$

which physically expresses the total illumination of the Earth's surface (by direct and scattered light) for a solar constant of unity. If the solar constant is πS , then the above expression must be multiplied by πS . To find the illumination by scattered light alone, it is sufficient to subtract the quantity $\cos \zeta e^{-\tau^* \sec \zeta}$ from the tabular value (for given τ^* and ζ).

Table X; Approximate Solutions of Schwarzschild and Eddington

Tables X.1 - X.20 contain a comparison of the approximate solutions of Schwarzschild (IV.18) and Eddington (IV.34) for the case $q = 0$, with the solution of the integral equation (II.33).

Column T gives the solution of the integral equation, column Sch the solution according to Schwarzschild, and column E the solution according to Eddington. Column I gives the solution when only single scattering is taken into consideration, calculated by the formula

$$\varphi_0(\tau) \approx \frac{1}{4} e^{-\tau^* \sec \zeta}.$$

Table XI; Approximate Solution According to Chandrasekhar

Tables XI.1 - XI.3 contain a comparison of the solution of the auxiliary integral equation (II.34) with the approximate solutions of the problem by the Chandrasekhar method. Column T gives the solution of the integral equation, and columns Ch₁ and Ch₂ the first and second approximations according to Chandrasekhar. Column Sch gives the approximate solution according to Schwarzschild, calculated by the formula

$$\omega(\tau) \approx \frac{\tau^* - \tau + \frac{1}{2}}{\tau^* + 1}.$$

The first approximation of Chandrasekhar was calculated by eq.(V.7) and the /52 second, by eq.(V.9).

Tables XI.4 - XI.8 contain a similar comparison for the solution of the integral equation (II.33) $\phi_0(\tau)$, which is repeated in column T.

The first approximation of Chandrasekhar, calculated by eq.(V.17), is given in column Ch₁, and the second approximation, calculated by eq.(V.21), in column Ch₂.

Table XIII; Haze Factors $\sigma(\tau, \theta)$

Tables XIII.1 - XIII.20 give the absolute haze factors as functions of the arguments τ and θ at an albedo of $q = 0$ of the earth's surface, calculated on the basis of eq.(VI.12) (a white object against a black background).

Equation (VI.14) can be used for transition to the case of an arbitrary albedo of the earth's surface.

TABLE I
FUNCTIONS $E_n(x)$ ($n = 1, 2, 3, 4$)

x	$E_1(x)$	$E_2(x)$	$E_3(x)$	$E_4(x)$
0.01	4.03792958	0.9496705	0.4902766	0.3283824
0.02	3.35470778	0.9131045	0.4809683	0.3235264
0.03	2.95911872	0.8816720	0.4719977	0.3187619
0.04	2.68126369	0.8535389	0.4633239	0.3140855
0.05	2.46789849	0.8278345	0.4549188	0.3094945
0.06	2.29530692	0.8040461	0.4467609	0.3049863
0.07	2.15083818	0.7818351	0.4388327	0.3005585
0.08	2.02694100	0.7609611	0.4311197	0.2962089
0.09	1.91874477	0.7412442	0.4236096	0.2919354
0.10	1.82292396	0.7225450	0.4162915	0.2877361
0.11	1.73710669	0.7047524	0.4091557	0.2836090
0.12	1.65954175	0.6877754	0.4021937	0.2795524
0.13	1.58889930	0.6715385	0.3953977	0.2755646
0.14	1.52414572	0.6559778	0.3887607	0.2716439
0.15	1.46446167	0.6410387	0.3822761	0.2677889
0.16	1.40918670	0.6266739	0.3759380	0.2639979
0.17	1.35778065	0.6128421	0.3697408	0.2602696
0.18	1.30979614	0.5995069	0.3636795	0.2566026
0.19	1.26485842	0.5866360	0.3577491	0.2529956
0.20	1.22265054	0.5742006	0.3519453	0.2494472
0.21	1.18290199	0.5621748	0.3462638	0.2459563
0.22	1.14538006	0.5505352	0.3407005	0.2425216
0.23	1.10988314	0.5392605	0.3352518	0.2391419
0.24	1.07623541	0.5283314	0.3299142	0.2358162
0.25	1.04428263	0.5177301	0.3246841	0.2325433
0.26	1.01388874	0.5074405	0.3195585	0.2293221
0.27	0.98493310	0.4974476	0.3145343	0.2261517
0.28	0.95730830	0.4877374	0.3096086	0.2230311
0.29	0.93091825	0.4782973	0.3047787	0.2199592
0.30	0.90567665	0.4691152	0.3000418	0.2169352
0.31	0.88150575	0.4601802	0.2953956	0.2139581
0.32	0.85833519	0.4514818	0.2908374	0.2110270
0.33	0.83610116	0.4430104	0.2863652	0.2081411
0.34	0.81474558	0.4347568	0.2819765	0.2052994
0.35	0.79421543	0.4267127	0.2776693	0.2025013
0.36	0.77446222	0.4188699	0.2734416	0.1997458
0.37	0.75544143	0.4112210	0.2692913	0.1970322
0.38	0.73711214	0.4037588	0.2652165	0.1943597
0.39	0.71943665	0.3964766	0.2612155	0.1917276
0.40	0.70238012	0.3893680	0.2572864	0.1891352
0.41	0.68591031	0.3824270	0.2534276	0.1865816
0.42	0.66999734	0.3756479	0.2496373	0.1840664
0.43	0.65461345	0.3690253	0.2459141	0.1815887
0.44	0.63973280	0.3625540	0.2422563	0.1791479
0.45	0.62533132	0.3562291	0.2386625	0.1767433
0.46	0.61138653	0.3500458	0.2351313	0.1743744
0.47	0.59787743	0.3439999	0.2316612	0.1720405
0.48	0.58478434	0.3380869	0.2282508	0.1697410

TABLE I (Cont'd)

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x	$E_1(x)$	$E_2(x)$	$E_3(x)$	$E_4(x)$
0.49	0.57208884	0.3323029	0.2248990	0.1674753
0.50	0.55977359	0.3266439	0.2216044	0.1652428
0.51	0.54782235	0.3211062	0.2183657	0.1630430
0.52	0.53621980	0.3156863	0.2151818	0.1608753
0.53	0.52495151	0.3103807	0.2120516	0.1587392
0.54	0.51400389	0.3051862	0.2089739	0.1566341
0.55	0.50336408	0.3000996	0.2059475	0.1545596
0.56	0.49301996	0.2951179	0.2029715	0.1525150
0.57	0.48290003	0.2902382	0.2000448	0.1505000
0.58	0.47317343	0.2854578	0.1971664	0.1485139
0.59	0.46361985	0.2807739	0.1943354	0.1465565
0.60	0.45437950	0.2761839	0.1915506	0.1446271

TABLE II.1
COEFFICIENTS P_{kn} AND Q_{kn} $h = 0.01$

m	$P_{0,m}$	$P_{60,m}$	$Q_{0,m}$	$Q_{60,m}$	m	$P_{0,m}$	$P_{60,m}$	$Q_{0,m}$	$Q_{60,m}$
0	0.050329	0.004590	0.0002267	0.0000230	31	0.008698	0.009440	0.0000434	0.0000474
1	36566	4684	1773	235	32	8171	9710	421	488
2	31433	4780	1539	240	33	8254	9993	411	502
3	28133	4880	1384	245	34	8044	10290	401	517
4	25704	4982	1268	250	35	0.007843	0.010001	0.0000390	0.0000533
5	0.023788	0.005087	0.0001174	0.0000255	36	7649	10929	381	550
6	22211	5195	1098	261	37	7462	11275	372	567
7	20874	5306	1034	267	38	7282	11640	362	585
8	19717	5420	977	272	39	7109	12026	354	605
9	18699	5538	926	277	40	0.006941	0.012435	0.0000315	0.0000626
10	0.017793	0.005059	0.0000883	0.0000284	41	6779	12871	338	648
11	16977	5781	842	291	42	6623	13335	329	671
12	16237	5913	806	296	43	6471	13832	323	695
13	15561	6046	772	304	44	6325	14365	320	723
14	14939	6183	742	306	45	0.006183	0.014939	0.0000307	0.0000752
15	0.014365	0.006325	0.0000714	0.0000317	46	6046	15561	301	784
16	13832	6471	688	325	47	5913	16237	295	818
17	13335	6623	662	333	48	5784	16977	288	855
18	12871	6779	640	340	49	5659	17793	282	897
19	12435	6941	618	349	50	0.005538	0.018099	0.0000276	0.0000943
20	0.012026	0.007109	0.0000598	0.0000357	51	5420	19717	270	995
21	11640	7282	579	366	52	5306	20874	264	1051
22	11275	7462	561	374	53	5195	22211	258	1122
23	10929	7649	543	384	54	5087	23788	254	1204
24	10601	7843	528	394	55	0.004982	0.025704	0.0000248	0.0001303
25	0.010290	0.008044	0.0000512	0.0000404	56	4880	28133	243	1430
26	9993	8254	497	414	57	4780	31433	238	1604
27	9710	8471	483	426	58	4684	36566	233	1884
28	9440	8698	469	437	59	4590	50329	229	2766
29	9182	8935	457	450	60	0.001498	0.050329	0.0000223	0.0002267
30	8935	9182	444	461					

TABLE II.2
COEFFICIENTS $P_{k,m}$ AND $Q_{k,m}$
 $h = 0.02$

m	$P_{0,m}$	$P_{30,m}$	$Q_{0,m}$	$Q_{30,m}$	m	$P_{0,m}$	$P_{30,m}$	$Q_{3,m}$	$Q_{30,m}$
0	0.036895	0.009274	0.0007696	0.00006934	16	0.016725	0.019704	0.0001658	0.0001989
1	59566	9660	5736	973	17	15887	20890	1575	2109
2	49493	10068	4821	1013	18	15111	22204	1499	2244
3	43085	10500	4220	1058	19	14391	23666	1427	2392
4	38416	10958	3773	1103	20	0.013720	0.025306	0.0001361	0.0002552
5	0.034770	0.011443	0.0003423	0.00001153	21	13094	27167	1299	2750
6	31797	11959	3134	1205	22	12508	29304	1240	2969
7	29304	12508	2892	1260	23	11959	31797	1187	3225
8	27167	13094	2684	1320	24	11443	34770	1136	3531
9	25306	13720	2502	1383	25	0.010958	0.038416	0.0001089	0.0003910
10	0.023666	0.014391	0.0002341	0.0001451	26	10500	43085	1042	4397
11	22204	15111	2197	1523	27	10068	49493	1000	5078
12	20890	15887	2069	1602	28	9660	59566	959	6177
13	19704	16725	1952	1687	29	9274	86895	921	9683
14	18622	17633	1845	1779	30	0.008908	0.086895	0.0000855	0.0007696
15	17633	18622	1748	1879					

TABLE III
SOLUTION OF THE AUXILIARY INTEGRAL EQUATION $\omega(\tau)$

τ	$\omega(\tau)$					τ	$\omega(\tau)$				
	$\tau^* = 0.2$	$\tau^* = 0.3$	$\tau^* = 0.4$	$\tau^* = 0.5$	$\tau^* = 0.6$		$\tau^* = 0.2$	$\tau^* = 0.3$	$\tau^* = 0.4$	$\tau^* = 0.5$	$\tau^* = 0.6$
0.0	0.6111	0.6419	0.6666	0.6874	0.7052	0.31					0.4933
0.01	0.5967	0.6286			0.6944	0.32					0.4877
0.02	0.5844	0.6175	0.6441	0.6664	0.6855	0.33					0.4815
0.03	0.5729	0.6072			0.6772	0.34					0.4753
0.04	0.5619	0.5973	0.6256	0.6492	0.6694	0.35					0.4692
0.05	0.5512	0.5878			0.6619	0.36					0.4630
0.06	0.5408	0.5786	0.6085	0.6233	0.6545	0.37					0.4568
0.07	0.5304	0.5605			0.6474	0.38					0.4505
0.08	0.5202	0.5695	0.5921	0.6181	0.6403	0.39					0.4443
0.09	0.5101	0.5517			0.6334	0.40					0.4380
0.10	0.5000	0.5440	0.5762	0.6035	0.6266	0.41					0.4317
0.11	0.4899	0.5343			0.6199	0.42					0.4254
0.12	0.4798	0.5257	0.5606	0.5891	0.6133	0.43					0.4191
0.13	0.4696	0.5171			0.6067	0.44					0.4127
0.14	0.4592	0.5085	0.5452	0.5751	0.6002	0.45					0.4063
0.15	0.4488	0.5000			0.5937	0.46					0.3998
0.16	0.4381	0.4915	0.5301	0.5612	0.5873	0.47					0.3933
0.17	0.4271	0.4829			0.5809	0.48					0.3867
0.18	0.4156	0.4743	0.5150	0.5474	0.5746	0.49					0.3801
0.19	0.4033	0.4657			0.5683	0.50					0.3734
0.20	0.3886	0.4570	0.5000	0.5338	0.5620	0.51					0.3666
0.21	0.4483				0.5557	0.52					0.3597
0.22	0.4395	0.4850	0.5203	0.5495	0.553						0.3526
0.23	0.4305				0.5432	0.51					0.345
0.24	0.4214	0.4699	0.5067	0.5370	0.55						0.3381
0.25	0.4122				0.5308	0.56					0.3306
0.26	0.4027	0.4547	0.4932	0.5247	0.57						0.3228
0.27	0.3928				0.5185	0.58					0.3145
0.28	0.3825	0.4394	0.4797	0.5123	0.59						0.3056
0.29	0.3714				0.5062	0.60					0.2948
0.30	0.3581	0.4238	0.4662	0.5000							

TABLE IV.1

SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\phi_0(\tau)$ $\tau^* = 0.2$

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τ	$\phi_0(\tau)$				τ	$\phi_0(\tau)$			
	$\zeta = 30^\circ$	$\zeta = 45^\circ$	$\zeta = 60^\circ$	$\zeta = 76^\circ$		$\zeta = 30^\circ$	$\zeta = 45^\circ$	$\zeta = 60^\circ$	$\zeta = 76^\circ$
0.00	0.2611	0.2493	0.2246	0.1547	0.11	0.3094	0.3022	0.2866	0.2366
0.01	0.2689	0.2572	0.2328	0.1620	0.12	0.3119	0.3052	0.2908	0.2442
0.02	0.2748	0.2634	0.2396	0.1705	0.13	0.3140	0.3080	0.2948	0.2520
0.03	0.2801	0.2690	0.2458	0.1777	0.14	0.3158	0.3104	0.2986	0.2597
0.04	0.2848	0.2742	0.2517	0.1850	0.15	0.3174	0.3126	0.3021	0.2675
0.05	0.2892	0.2790	0.2572	0.1921	0.16	0.3185	0.3144	0.3054	0.2753
0.06	0.2933	0.2835	0.2626	0.1994	0.17	0.3191	0.3157	0.3081	0.2829
0.07	0.2970	0.2877	0.2678	0.2066	0.18	0.3191	0.3164	0.3104	0.2903
0.08	0.3005	0.2917	0.2727	0.2140	0.19	0.3184	0.3163	0.3119	0.2973
0.09	0.3037	0.2954	0.2775	0.2215	0.20	0.3153	0.3140	0.3112	0.3023
0.10	0.3067	0.2989	0.2821	0.2290					

TABLE IV.2

SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\phi_0(\tau)$ $\tau^* = 0.3$

τ	$\phi_0(\tau)$				τ	$\phi_0(\tau)$			
	$\zeta = 30^\circ$	$\zeta = 45^\circ$	$\zeta = 60^\circ$	$\zeta = 76^\circ$		$\zeta = 30^\circ$	$\zeta = 45^\circ$	$\zeta = 60^\circ$	$\zeta = 76^\circ$
0.00	0.2577	0.2410	0.2075	0.1229	0.16	0.3274	0.3158	0.2911	0.2177
0.01	0.2657	0.2490	0.2153	0.1293	0.17	0.3300	0.3188	0.2952	0.2243
0.02	0.2720	0.2554	0.2218	0.1350	0.18	0.3323	0.3218	0.2992	0.2311
0.03	0.2777	0.2612	0.2278	0.1406	0.19	0.3345	0.3245	0.3032	0.2380
0.04	0.2828	0.2666	0.2334	0.1462	0.20	0.3365	0.3271	0.3069	0.2449
0.05	0.2877	0.2717	0.2390	0.1517	0.21	0.3383	0.3294	0.3106	0.2520
0.06	0.2923	0.2765	0.2443	0.1573	0.22	0.3398	0.3316	0.3140	0.2592
0.07	0.2967	0.2812	0.2494	0.1630	0.23	0.3411	0.3335	0.3173	0.2665
0.08	0.3008	0.2857	0.2545	0.1686	0.24	0.3421	0.3352	0.3204	0.2738
0.09	0.3048	0.2900	0.2594	0.1744	0.25	0.3427	0.3366	0.3232	0.2811
0.10	0.3085	0.2941	0.2642	0.1803	0.26	0.3430	0.3376	0.3258	0.2884
0.11	0.3121	0.2980	0.2689	0.1863	0.27	0.3429	0.3381	0.3279	0.2956
0.12	0.3155	0.3019	0.2734	0.1923	0.28	0.3421	0.3381	0.3295	0.3026
0.13	0.3187	0.3056	0.2780	0.1985	0.29	0.3404	0.3372	0.3302	0.3091
0.14	0.3218	0.3091	0.2825	0.2048	0.30	0.3364	0.3339	0.3288	0.3137
0.15	0.3247	0.3126	0.2868	0.2112					

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TABLE IV.3
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_0(\tau)$

$\tau^* = 0.4$

τ	$\varphi_0(\tau)$				τ	$\varphi_0(\tau)$			
	$\zeta = 30^\circ$	$\zeta = 45^\circ$	$\zeta = 60^\circ$	$\zeta = 76^\circ$		$\zeta = 30^\circ$	$\zeta = 45^\circ$	$\zeta = 60^\circ$	$\zeta = 76^\circ$
0.00	0.2528	0.2319	0.1915	0.0997	0.22	0.3455	0.3297	0.2969	0.2053
0.02	0.2674	0.2462	0.2049	0.1092	0.24	0.3502	0.3355	0.3047	0.2171
0.04	0.2786	0.2574	0.2159	0.1178	0.26	0.3545	0.3408	0.3122	0.2294
0.06	0.2885	0.2676	0.2262	0.1264	0.28	0.3580	0.3456	0.3193	0.2422
0.08	0.2977	0.2770	0.2360	0.1351	0.30	0.3609	0.3497	0.3260	0.2555
0.10	0.3060	0.2858	0.2454	0.1441	0.32	0.3630	0.3531	0.3322	0.2692
0.12	0.3139	0.2942	0.2545	0.1533	0.34	0.3641	0.3556	0.3376	0.2833
0.14	0.3212	0.3021	0.2635	0.1629	0.36	0.3638	0.3568	0.3420	0.2974
0.16	0.3280	0.3096	0.2722	0.1728	0.38	0.3617	0.3562	0.3448	0.3111
0.18	0.3343	0.3167	0.2806	0.1832	0.40	0.3546	0.3508	0.3429	0.3215
0.20	0.3402	0.3234	0.2889	0.1940					

TABLE IV.4
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_0(\tau)$

$\tau^* = 0.5$

τ	$\varphi_0(\tau)$				τ	$\varphi_0(\tau)$			
	$\zeta = 30^\circ$	$\zeta = 45^\circ$	$\zeta = 60^\circ$	$\zeta = 76^\circ$		$\zeta = 30^\circ$	$\zeta = 45^\circ$	$\zeta = 60^\circ$	$\zeta = 76^\circ$
0.00	0.2472	0.2228	0.1771	0.0830	0.26	0.3566	0.3356	0.2932	0.1833
0.02	0.2618	0.2367	0.1896	0.0905	0.28	0.3617	0.3416	0.3008	0.1932
0.04	0.2732	0.2478	0.1999	0.0973	0.30	0.3664	0.3473	0.3083	0.2036
0.06	0.2835	0.2580	0.2095	0.1040	0.32	0.3707	0.3526	0.3155	0.2145
0.08	0.2929	0.2674	0.2187	0.1108	0.34	0.3745	0.3574	0.3225	0.2260
0.10	0.3017	0.2763	0.2276	0.1177	0.36	0.3776	0.3618	0.3292	0.2379
0.12	0.3099	0.2849	0.2363	0.1248	0.38	0.3802	0.3657	0.3357	0.2504
0.14	0.3178	0.2931	0.2449	0.1322	0.40	0.3821	0.3690	0.3417	0.2633
0.16	0.3252	0.3009	0.2532	0.1398	0.42	0.3832	0.3715	0.3471	0.2766
0.18	0.3322	0.3084	0.2615	0.1478	0.44	0.3834	0.3731	0.3518	0.2904
0.20	0.3389	0.3157	0.2696	0.1561	0.46	0.3822	0.3735	0.3556	0.3041
0.22	0.3452	0.3226	0.2776	0.1647	0.48	0.3790	0.3720	0.3576	0.3173
0.24	0.3511	0.3293	0.2855	0.1738	0.50	0.3707	0.3654	0.3548	0.3274

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TABLE VII.1
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.2; \zeta = 30^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2611	0.3093	0.3589	0.4102	0.6928
0.01	0.2689	0.3169	0.3644	0.4144	0.6901
0.02	0.2748	0.3208	0.3683	0.4173	0.6874
0.03	0.2801	0.3252	0.3718	0.4198	0.6846
0.04	0.2848	0.3291	0.3747	0.4218	0.6815
0.05	0.2892	0.3326	0.3774	0.4236	0.6783
0.06	0.2933	0.3359	0.3798	0.4252	0.6750
0.07	0.2970	0.3388	0.3819	0.4263	0.6715
0.08	0.3005	0.3415	0.3838	0.4274	0.6678
0.09	0.3037	0.3439	0.3853	0.4281	0.6638
0.10	0.3067	0.3461	0.3867	0.4286	0.6597
0.11	0.3094	0.3480	0.3878	0.4289	0.6552
0.12	0.3119	0.3497	0.3887	0.4289	0.6506
0.13	0.3140	0.3510	0.3891	0.4285	0.6455
0.14	0.3159	0.3521	0.3894	0.4279	0.6401
0.15	0.3174	0.3528	0.3892	0.4268	0.6342
0.16	0.3185	0.3530	0.3886	0.4253	0.6278
0.17	0.3191	0.3528	0.3874	0.4232	0.6206
0.18	0.3191	0.3518	0.3856	0.4204	0.6125
0.19	0.3184	0.3502	0.3829	0.4167	0.6031
0.20	0.3153	0.3459	0.3775	0.4100	0.5896

TABLE VII.2
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.2; \zeta = 45^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2493	0.2877	0.3274	0.3682	0.5937
0.01	0.2572	0.2947	0.3334	0.3732	0.5933
0.02	0.2634	0.3001	0.3380	0.3770	0.5926
0.03	0.2690	0.3050	0.3421	0.3805	0.5917
0.04	0.2742	0.3095	0.3459	0.3835	0.5907
0.05	0.2790	0.3136	0.3494	0.3862	0.5895
0.06	0.2835	0.3175	0.3525	0.3887	0.5881
0.07	0.2877	0.3210	0.3554	0.3909	0.5865
0.08	0.2917	0.3244	0.3581	0.3929	0.5847
0.09	0.2954	0.3275	0.3605	0.3946	0.5827
0.10	0.2989	0.3303	0.3627	0.3962	0.5805
0.11	0.3022	0.3330	0.3647	0.3975	0.5781
0.12	0.3052	0.3354	0.3665	0.3985	0.5754
0.13	0.3080	0.3375	0.3679	0.3993	0.5725
0.14	0.3104	0.3393	0.3690	0.3997	0.5691
0.15	0.3126	0.3408	0.3699	0.3999	0.5654
0.16	0.3144	0.3419	0.3703	0.3996	0.5612
0.17	0.3157	0.3425	0.3702	0.3988	0.5563
0.18	0.3164	0.3425	0.3695	0.3973	0.5505
0.19	0.3163	0.3417	0.3678	0.3948	0.5435
0.20	0.3140	0.3384	0.3636	0.3896	0.5329

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TABLE VII.3
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.2; \zeta = 60^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2246	0.2505	0.2771	0.3046	0.4562
0.01	0.2328	0.2580	0.2840	0.3109	0.4589
0.02	0.2396	0.2643	0.2898	0.3161	0.4610
0.03	0.2458	0.2700	0.2950	0.3208	0.4628
0.04	0.2517	0.2755	0.3000	0.3252	0.4646
0.05	0.2572	0.2805	0.3045	0.3293	0.4660
0.06	0.2626	0.2855	0.3090	0.3334	0.4675
0.07	0.2678	0.2902	0.3133	0.3372	0.4688
0.08	0.2727	0.2947	0.3174	0.3408	0.4698
0.09	0.2775	0.2991	0.3213	0.3442	0.4707
0.10	0.2821	0.3032	0.3250	0.3475	0.4715
0.11	0.2866	0.3073	0.3287	0.3507	0.4722
0.12	0.2908	0.3111	0.3320	0.3536	0.4726
0.13	0.2948	0.3147	0.3351	0.3562	0.4727
0.14	0.2986	0.3180	0.3380	0.3587	0.4726
0.15	0.3021	0.3211	0.3406	0.3608	0.4721
0.16	0.3054	0.3239	0.3430	0.3627	0.4714
0.17	0.3081	0.3262	0.3448	0.3640	0.4699
0.18	0.3104	0.3280	0.3461	0.3648	0.4678
0.19	0.3119	0.3290	0.3465	0.3647	0.4647
0.20	0.3112	0.3276	0.3446	0.3620	0.4584

TABLE VII.4
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.2; \zeta = 76^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.1547	0.1654	0.1764	0.1877	0.2503
0.01	0.1630	0.1734	0.1842	0.1952	0.2563
0.02	0.1705	0.1807	0.1912	0.2021	0.2619
0.03	0.1777	0.1877	0.1980	0.2086	0.2673
0.04	0.1850	0.1918	0.2049	0.2153	0.2729
0.05	0.1921	0.2017	0.2116	0.2219	0.2783
0.06	0.1994	0.2088	0.2186	0.2286	0.2840
0.07	0.2067	0.2160	0.2255	0.2353	0.2897
0.08	0.2140	0.2231	0.2324	0.2421	0.2954
0.09	0.2215	0.2304	0.2396	0.2491	0.3013
0.10	0.2290	0.2377	0.2467	0.2560	0.3072
0.11	0.2366	0.2451	0.2540	0.2631	0.3132
0.12	0.2442	0.2526	0.2612	0.2701	0.3192
0.13	0.2520	0.2602	0.2686	0.2774	0.3254
0.14	0.2597	0.2677	0.2760	0.2845	0.3315
0.15	0.2675	0.2753	0.2834	0.2917	0.3377
0.16	0.2753	0.2829	0.2908	0.2990	0.3438
0.17	0.2829	0.2904	0.2980	0.3060	0.3497
0.18	0.2903	0.2976	0.3050	0.3127	0.3553
0.19	0.2973	0.3043	0.3116	0.3191	0.3604
0.20	0.3023	0.3091	0.3161	0.3233	0.3631

TABLE VII.5
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.3; \zeta = 30^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2577	0.3060	0.3564	0.4090	0.7110
0.01	0.2657	0.3130	0.3624	0.4139	0.7098
0.02	0.2720	0.3185	0.3670	0.4176	0.7082
0.03	0.2777	0.3234	0.3711	0.4208	0.7066
0.04	0.2828	0.3278	0.3796	0.4237	0.7048
0.05	0.2877	0.3320	0.3827	0.4263	0.7030
0.06	0.2923	0.3358	0.3813	0.4287	0.7010
0.07	0.2967	0.3395	0.3843	0.4309	0.6990
0.08	0.3008	0.3430	0.3870	0.4329	0.6968
0.09	0.3048	0.3463	0.3896	0.4348	0.6945
0.10	0.3085	0.3494	0.3920	0.4365	0.6921
0.11	0.3121	0.3523	0.3943	0.4380	0.6895
0.12	0.3155	0.3550	0.3963	0.4394	0.6868
0.13	0.3187	0.3576	0.3982	0.4406	0.6840
0.14	0.3218	0.3600	0.4000	0.4417	0.6810
0.15	0.3247	0.3623	0.4016	0.4426	0.6779
0.16	0.3274	0.3644	0.4030	0.4433	0.6746
0.17	0.3300	0.3663	0.4042	0.4438	0.6711
0.18	0.3324	0.3681	0.4058	0.4442	0.6675
0.19	0.3345	0.3696	0.4061	0.4443	0.6635
0.20	0.3365	0.3709	0.4067	0.4442	0.6594
0.21	0.3383	0.3720	0.4072	0.4440	0.6550
0.22	0.3398	0.3729	0.4074	0.4434	0.6502
0.23	0.3411	0.3735	0.4073	0.4426	0.6452
0.24	0.3421	0.3738	0.4069	0.4414	0.6398
0.25	0.3427	0.3738	0.4061	0.4399	0.6339
0.26	0.3430	0.3734	0.4050	0.4380	0.6275
0.27	0.3429	0.3724	0.4039	0.4355	0.6204
0.28	0.3421	0.3709	0.4019	0.4323	0.6123
0.29	0.3404	0.3684	0.3976	0.4280	0.6028
0.30	0.3364	0.3633	0.3914	0.4208	0.5893

TABLE VII.6
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.3; \zeta = 45^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2410	0.2792	0.3190	0.3606	0.5993
0.01	0.2490	0.2864	0.3254	0.3661	0.5998
0.02	0.2554	0.2921	0.3304	0.3704	0.6001
0.03	0.2612	0.2973	0.3350	0.3743	0.6001
0.04	0.2666	0.3021	0.3392	0.3778	0.6000
0.05	0.2717	0.3065	0.3431	0.3812	0.5998
0.06	0.2765	0.3110	0.3468	0.3843	0.5995
0.07	0.2812	0.3151	0.3504	0.3873	0.5991
0.08	0.2857	0.3190	0.3538	0.3901	0.5985

TABLE VII.6 (Cont'd)

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τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.09	0.2900	0.3228	0.3570	0.3927	0.5979
0.10	0.2941	0.3264	0.3601	0.3952	0.5972
0.11	0.2980	0.3298	0.3630	0.3976	0.5963
0.12	0.3019	0.3332	0.3658	0.3998	0.5953
0.13	0.3056	0.3364	0.3684	0.4019	0.5942
0.14	0.3091	0.3394	0.3709	0.4039	0.5930
0.15	0.3126	0.3423	0.3733	0.4057	0.5917
0.16	0.3158	0.3450	0.3755	0.4073	0.5901
0.17	0.3188	0.3476	0.3775	0.4088	0.5884
0.18	0.3218	0.3500	0.3794	0.4101	0.5865
0.19	0.3245	0.3522	0.3811	0.4113	0.5845
0.20	0.3271	0.3542	0.3826	0.4122	0.5822
0.21	0.3294	0.3561	0.3839	0.4130	0.5797
0.22	0.3316	0.3578	0.3850	0.4135	0.5769
0.23	0.3335	0.3592	0.3859	0.4137	0.5738
0.24	0.3352	0.3603	0.3864	0.4137	0.5704
0.25	0.3366	0.3611	0.3866	0.4133	0.5666
0.26	0.3376	0.3615	0.3865	0.4126	0.5623
0.27	0.3381	0.3615	0.3859	0.4113	0.5574
0.28	0.3381	0.3608	0.3846	0.4094	0.5516
0.29	0.3372	0.3593	0.3823	0.4064	0.5415
0.30	0.3339	0.3652	0.3774	0.4006	0.5338

TABLE VII.7
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$ $\tau^* = 0.3; \zeta = 60^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2075	0.2327	0.2589	0.2863	0.4438
0.01	0.2153	0.2400	0.2657	0.2925	0.4467
0.02	0.2218	0.2460	0.2712	0.2976	0.4490
0.03	0.2278	0.2516	0.2764	0.3024	0.4512
0.04	0.2334	0.2565	0.2813	0.3068	0.4533
0.05	0.2390	0.2620	0.2860	0.3112	0.4553
0.06	0.2443	0.2670	0.2906	0.3153	0.4572
0.07	0.2494	0.2718	0.2950	0.3194	0.4590
0.08	0.2545	0.2765	0.2994	0.3234	0.4608
0.09	0.2594	0.2810	0.3036	0.3271	0.4624
0.10	0.2642	0.2855	0.3077	0.3309	0.4641
0.11	0.2689	0.2899	0.3117	0.3346	0.4656
0.12	0.2736	0.2942	0.3157	0.3381	0.4671
0.13	0.2780	0.2983	0.3195	0.3416	0.4683
0.14	0.2825	0.3024	0.3233	0.3450	0.4697
0.15	0.2868	0.3064	0.3269	0.3482	0.4709
0.16	0.2911	0.3103	0.3304	0.3514	0.4720
0.17	0.2952	0.3142	0.3339	0.3545	0.4730
0.18	0.2992	0.3178	0.3372	0.3575	0.4738
0.19	0.3032	0.3214	0.3405	0.3604	0.4746

TABLE VII.7 (Cont'd)

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τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.20	0.3069	0.3248	0.3436	0.3631	0.4752
0.21	0.3106	0.3282	0.3465	0.3656	0.4756
0.22	0.3140	0.3313	0.3493	0.3680	0.4758
0.23	0.3173	0.3342	0.3518	0.3702	0.4758
0.24	0.3204	0.3370	0.3542	0.3722	0.4755
0.25	0.3232	0.3394	0.3563	0.3738	0.4750
0.26	0.3258	0.3416	0.3580	0.3752	0.4740
0.27	0.3279	0.3433	0.3594	0.3762	0.4725
0.28	0.3295	0.3445	0.3602	0.3765	0.4703
0.29	0.3302	0.3448	0.3602	0.3759	0.4670
0.30	0.3288	0.3428	0.3574	0.3727	0.4605

TABLE VII.8

SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$ $\tau^* = 0.3; \zeta = 76^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.1229	0.1328	0.1431	0.1539	0.2159
0.01	0.1293	0.1390	0.1492	0.1597	0.2204
0.02	0.1350	0.1446	0.1545	0.1649	0.2246
0.03	0.1406	0.1500	0.1598	0.1700	0.2286
0.04	0.1462	0.1554	0.1650	0.1751	0.2328
0.05	0.1517	0.1608	0.1703	0.1802	0.2369
0.06	0.1573	0.1662	0.1756	0.1853	0.2412
0.07	0.1630	0.1718	0.1809	0.1905	0.2455
0.08	0.1686	0.1773	0.1863	0.1957	0.2499
0.09	0.1744	0.1830	0.1918	0.2011	0.2544
0.10	0.1803	0.1870	0.1974	0.2036	0.2590
0.11	0.1863	0.1945	0.2031	0.2121	0.2637
0.12	0.1923	0.2004	0.2089	0.2178	0.2685
0.13	0.1985	0.2065	0.2148	0.2235	0.2735
0.14	0.2048	0.2127	0.2208	0.2294	0.2785
0.15	0.2112	0.2189	0.2270	0.2354	0.2837
0.16	0.2177	0.2253	0.2332	0.2415	0.2889
0.17	0.2243	0.2318	0.2336	0.2477	0.2943
0.18	0.2311	0.2384	0.2460	0.2540	0.2998
0.19	0.2380	0.2452	0.2526	0.2605	0.3055
0.20	0.2449	0.2520	0.2594	0.2670	0.3112
0.21	0.2520	0.2589	0.2662	0.2737	0.3170
0.22	0.2592	0.2660	0.2730	0.2804	0.3229
0.23	0.2665	0.2731	0.2800	0.2873	0.3289
0.24	0.2738	0.2803	0.2871	0.2942	0.3349
0.25	0.2811	0.2875	0.2941	0.3010	0.3408
0.26	0.2884	0.2946	0.3011	0.3079	0.3468
0.27	0.2956	0.3017	0.3080	0.3146	0.3526
0.28	0.3026	0.3086	0.3147	0.3212	0.3581
0.29	0.3091	0.3148	0.3238	0.3271	0.3629
0.30	0.3137	0.3192	0.3250	0.3310	0.3656

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TABLE VII.9
 SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$
 $\tau^* = 0.4; \zeta = 30^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2528	0.3008	0.3514	0.4048	0.7228
0.02	0.2674	0.3138	0.3627	0.4142	0.7215
0.04	0.2786	0.3237	0.3711	0.4212	0.7197
0.06	0.2885	0.3323	0.3785	0.4272	0.7175
0.08	0.2977	0.3403	0.3853	0.4327	0.7151
0.10	0.3061	0.3476	0.3913	0.4375	0.7123
0.12	0.3139	0.3543	0.3968	0.4417	0.7091
0.14	0.3212	0.3605	0.4019	0.4455	0.7056
0.16	0.3280	0.3662	0.4064	0.4488	0.7017
0.18	0.3343	0.3714	0.4105	0.4517	0.6974
0.20	0.3402	0.3762	0.4142	0.4542	0.6927
0.22	0.3455	0.3804	0.4172	0.4561	0.6874
0.24	0.3502	0.3840	0.4197	0.4573	0.6815
0.26	0.3545	0.3872	0.4218	0.4582	0.6751
0.28	0.3580	0.3896	0.4230	0.4582	0.6678
0.30	0.3609	0.3914	0.4236	0.4575	0.6597
0.32	0.3630	0.3924	0.4233	0.4560	0.6506
0.34	0.3641	0.3923	0.4220	0.4534	0.6401
0.36	0.3639	0.3909	0.4193	0.4492	0.6278
0.38	0.3617	0.3873	0.4143	0.4428	0.6126
0.40	0.3546	0.3786	0.4039	0.4306	0.5896

TABLE VII.10
 SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$
 $\tau^* = 0.4; \zeta = 45^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2319	0.2695	0.3091	0.3509	0.6000
0.02	0.2462	0.2825	0.3208	0.3612	0.6019
0.04	0.2574	0.2927	0.3299	0.3691	0.6029
0.06	0.2676	0.3019	0.3381	0.3763	0.6036
0.08	0.2770	0.3104	0.3456	0.3827	0.6040
0.10	0.2858	0.3183	0.3525	0.3887	0.6040
0.12	0.2942	0.3258	0.3591	0.3943	0.6038
0.14	0.3021	0.3329	0.3653	0.3995	0.6032
0.16	0.3096	0.3395	0.3710	0.4043	0.6023
0.18	0.3167	0.3458	0.3764	0.4087	0.6011
0.20	0.3234	0.3516	0.3813	0.4127	0.5995
0.22	0.3297	0.3571	0.3859	0.4163	0.5975
0.24	0.3355	0.3620	0.3899	0.4194	0.5950
0.26	0.3408	0.3665	0.3935	0.4220	0.5919
0.28	0.3456	0.3704	0.3965	0.4241	0.5883
0.30	0.3497	0.3736	0.3988	0.4254	0.5838
0.32	0.3531	0.3761	0.4004	0.4259	0.5784
0.34	0.3556	0.3777	0.4010	0.4255	0.5718
0.36	0.3568	0.3779	0.4002	0.4237	0.5635
0.38	0.3562	0.3763	0.3971	0.4198	0.5527
0.40	0.3508	0.3696	0.3894	0.4103	0.5349

TABLE VII.11
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.4; \zeta = 60^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.1915	0.2159	0.2416	0.2687	0.4303
0.02	0.2049	0.2285	0.2533	0.2795	0.4356
0.04	0.2159	0.2388	0.2629	0.2884	0.4400
0.06	0.2262	0.2485	0.2719	0.2967	0.4441
0.08	0.2360	0.2577	0.2805	0.3046	0.4480
0.10	0.2454	0.2665	0.2887	0.3121	0.4518
0.12	0.2545	0.2750	0.2966	0.3194	0.4553
0.14	0.2635	0.2835	0.3045	0.3266	0.4588
0.16	0.2722	0.2916	0.3120	0.3336	0.4620
0.18	0.2806	0.2994	0.3193	0.3402	0.4650
0.20	0.2889	0.3072	0.3265	0.3468	0.4680
0.22	0.2969	0.3146	0.3333	0.3531	0.4706
0.24	0.3047	0.3219	0.3400	0.3591	0.4730
0.25	0.3122	0.3288	0.3452	0.3649	0.4751
0.28	0.3193	0.3354	0.3523	0.3702	0.4767
0.30	0.3260	0.3415	0.3578	0.3751	0.4778
0.32	0.3322	0.3471	0.3628	0.3794	0.4783
0.34	0.3376	0.3519	0.3670	0.3829	0.4778
0.36	0.3420	0.3557	0.3702	0.3854	0.4761
0.38	0.3448	0.3578	0.3715	0.3860	0.4723
0.40	0.3429	0.3551	0.3679	0.3815	0.4623

TABLE VII.12
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.4; \zeta = 76^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.0997	0.1090	0.1188	0.1291	0.1907
0.02	0.1092	0.1182	0.1277	0.1376	0.1972
0.04	0.1178	0.1265	0.1357	0.1454	0.2032
0.06	0.1264	0.1349	0.1438	0.1533	0.2095
0.08	0.1351	0.1434	0.1521	0.1612	0.2159
0.10	0.1441	0.1521	0.1606	0.1695	0.2228
0.12	0.1533	0.1611	0.1694	0.1781	0.2298
0.14	0.1629	0.1705	0.1785	0.1870	0.2374
0.16	0.1728	0.1802	0.1880	0.1962	0.2452
0.18	0.1832	0.1904	0.1980	0.2059	0.2535
0.20	0.1940	0.2010	0.2083	0.2161	0.2623
0.22	0.2053	0.2121	0.2192	0.2267	0.2715
0.24	0.2171	0.2237	0.2306	0.2379	0.2813
0.26	0.2294	0.2357	0.2424	0.2495	0.2915
0.28	0.2422	0.2483	0.2547	0.2616	0.3022
0.30	0.2555	0.2614	0.2676	0.2742	0.3134
0.32	0.2692	0.2749	0.2809	0.2872	0.3249
0.34	0.2833	0.2883	0.2945	0.3006	0.3368
0.36	0.2974	0.3026	0.3081	0.3139	0.3485
0.38	0.3111	0.3161	0.3213	0.3268	0.3597
0.40	0.3215	0.3262	0.3311	0.3362	0.3670

TABLE VII.13
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_a(\tau)$
 $\tau^* = 0.5; \zeta = 30^\circ$

τ	$\varphi_a(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2472	0.2946	0.3450	0.3987	0.7296
0.02	0.2619	0.3079	0.3568	0.4088	0.7296
0.04	0.2733	0.3181	0.3657	0.4164	0.7290
0.06	0.2835	0.3272	0.3737	0.4231	0.7280
0.08	0.2929	0.3356	0.3809	0.4292	0.7268
0.10	0.3017	0.3433	0.3876	0.4347	0.7253
0.12	0.3099	0.3506	0.3938	0.4398	0.7234
0.14	0.3178	0.3575	0.3997	0.4446	0.7214
0.16	0.3252	0.3639	0.4051	0.4489	0.7191
0.18	0.3322	0.3700	0.4101	0.4529	0.7164
0.20	0.3389	0.3757	0.4149	0.4566	0.7136
0.22	0.3452	0.3790	0.4193	0.4599	0.7104
0.24	0.3511	0.3840	0.4232	0.4628	0.7068
0.26	0.3560	0.3887	0.4268	0.4653	0.7028
0.28	0.3617	0.3929	0.4300	0.4675	0.6984
0.30	0.3664	0.3969	0.4328	0.4692	0.6935
0.32	0.3707	0.4001	0.4351	0.5705	0.6883
0.34	0.3745	0.4030	0.4370	0.4713	0.6825
0.36	0.3776	0.4052	0.4381	0.4713	0.6759
0.38	0.3802	0.4069	0.4387	0.4708	0.6686
0.40	0.3821	0.4079	0.4385	0.4695	0.6604
0.42	0.3832	0.4080	0.4376	0.4674	0.6512
0.44	0.3834	0.4072	0.4356	0.4642	0.6408
0.46	0.3822	0.4050	0.4321	0.4595	0.6284
0.48	0.3790	0.4007	0.4265	0.4526	0.6132
0.50	0.3707	0.3910	0.4152	0.4396	0.5901

TABLE VII.14
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_a(\tau)$

 $\tau^* = 0.5; \zeta = 45^\circ$

τ	$\varphi_a(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2228	0.2596	0.2988	0.3405	0.5974
0.02	0.2367	0.2724	0.3104	0.3508	0.5999
0.04	0.2479	0.2827	0.3197	0.3591	0.6018
0.06	0.2580	0.2919	0.3280	0.3664	0.6032
0.08	0.2674	0.3005	0.3357	0.3732	0.6043
0.10	0.2763	0.3086	0.3430	0.3796	0.6052
0.12	0.2849	0.3165	0.3500	0.3851	0.6060
0.14	0.2931	0.3239	0.3567	0.3916	0.6065
0.16	0.3009	0.3310	0.3629	0.3970	0.6068
0.18	0.3085	0.3378	0.3690	0.4022	0.6069
0.20	0.3157	0.3443	0.3747	0.4071	0.6067

TABLE VII.14 (Cont'd)

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τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.22	0.3226	0.3505	0.3801	0.4117	0.6062
0.24	0.3293	0.3565	0.3853	0.4161	0.6055
0.26	0.3356	0.3620	0.3901	0.4200	0.6044
0.28	0.3416	0.3673	0.3946	0.4237	0.6031
0.30	0.3473	0.3723	0.3988	0.4271	0.6014
0.32	0.3526	0.3769	0.4026	0.4301	0.5993
0.34	0.3574	0.3809	0.4059	0.4325	0.5966
0.36	0.3618	0.3846	0.4088	0.4346	0.5934
0.38	0.3657	0.3877	0.4111	0.4360	0.5896
0.40	0.3690	0.3903	0.4128	0.4369	0.5851
0.42	0.3715	0.3920	0.4137	0.4369	0.5796
0.44	0.3731	0.3928	0.4136	0.4359	0.5729
0.45	0.3735	0.3923	0.4123	0.4336	0.5647
0.48	0.3720	0.3899	0.4089	0.4291	0.5538
0.50	0.3654	0.3822	0.4000	0.4189	0.5358

TABLE VII.15

SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$$\tau^* = 0.5; \zeta = 60^\circ$$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.1771	0.2007	0.2257	0.2524	0.4168
0.02	0.1896	0.2124	0.2367	0.2626	0.4220
0.04	0.1999	0.2221	0.2458	0.2710	0.4263
0.06	0.2095	0.2312	0.2543	0.2789	0.4304
0.08	0.2187	0.2399	0.2624	0.2864	0.4343
0.10	0.2276	0.2483	0.2703	0.2937	0.4381
0.12	0.2363	0.2565	0.2780	0.3009	0.4418
0.14	0.2449	0.2646	0.2855	0.3079	0.4454
0.16	0.2532	0.2725	0.2929	0.3147	0.4490
0.18	0.2615	0.2802	0.3002	0.3214	0.4524
0.20	0.2696	0.2879	0.3074	0.3281	0.4558
0.22	0.2776	0.2954	0.3144	0.3346	0.4591
0.24	0.2855	0.3029	0.3213	0.3410	0.4622
0.26	0.2932	0.3102	0.3282	0.3473	0.4653
0.28	0.3008	0.3173	0.3348	0.3534	0.4682
0.30	0.3083	0.3242	0.3412	0.3593	0.4709
0.32	0.3155	0.3310	0.3475	0.3651	0.4734
0.34	0.3225	0.3376	0.3536	0.3706	0.4756
0.36	0.3292	0.3438	0.3593	0.3758	0.4775
0.38	0.3357	0.3498	0.3647	0.3807	0.4790
0.40	0.3417	0.3553	0.3697	0.3851	0.4800
0.42	0.3471	0.3602	0.3741	0.3890	0.4803
0.44	0.3518	0.3644	0.3778	0.3920	0.4797
0.46	0.3556	0.3676	0.3804	0.3940	0.4779
0.48	0.3576	0.3690	0.3812	0.3941	0.4739
0.50	0.3548	0.3645	0.3769	0.3891	0.4638

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TABLE VII.16
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.5; \zeta = 76^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.0830	0.0918	0.1011	0.1111	0.1724
0.02	0.0905	0.0990	0.1081	0.1178	0.1772
0.04	0.0973	0.1056	0.1144	0.1238	0.1818
0.06	0.1040	0.1121	0.1207	0.1299	0.1864
0.08	0.1108	0.1187	0.1271	0.1360	0.1912
0.10	0.1177	0.1254	0.1336	0.1423	0.1962
0.12	0.1248	0.1323	0.1403	0.1489	0.2014
0.14	0.1322	0.1395	0.1473	0.1557	0.2070
0.16	0.1398	0.1470	0.1546	0.1628	0.2128
0.18	0.1478	0.1548	0.1622	0.1701	0.2190
0.20	0.1561	0.1629	0.1702	0.1779	0.2255
0.22	0.1647	0.1714	0.1784	0.1860	0.2324
0.24	0.1738	0.1803	0.1872	0.1945	0.2397
0.26	0.1833	0.1896	0.1963	0.2034	0.2474
0.28	0.1932	0.1994	0.2059	0.2128	0.2556
0.30	0.2036	0.2096	0.2150	0.2227	0.2643
0.32	0.2145	0.2203	0.2269	0.2330	0.2734
0.34	0.2260	0.2316	0.2375	0.2439	0.2830
0.36	0.2379	0.2433	0.2491	0.2552	0.2932
0.38	0.2504	0.2556	0.2612	0.2671	0.3034
0.40	0.2633	0.2684	0.2738	0.2795	0.3149
0.42	0.2766	0.2815	0.2867	0.2922	0.3263
0.44	0.2904	0.2950	0.3000	0.3053	0.3381
0.46	0.3041	0.3086	0.3134	0.3184	0.3498
0.48	0.3174	0.3217	0.3262	0.3311	0.3608
0.50	0.3274	0.3314	0.3357	0.3402	0.3681

TABLE VII.17
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.6; \zeta = 30^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2411	0.2878	0.3378	0.3915	0.7330
0.02	0.2556	0.3010	0.3496	0.4018	0.7338
0.04	0.2670	0.3113	0.3588	0.4097	0.7340
0.06	0.2773	0.3206	0.3670	0.4169	0.7339
0.08	0.2869	0.3293	0.3747	0.4235	0.7336
0.10	0.2958	0.3373	0.3817	0.4294	0.7329
0.12	0.3044	0.3450	0.3885	0.4352	0.7322
0.14	0.3125	0.3522	0.3948	0.4405	0.7312
0.16	0.3202	0.3591	0.4007	0.4454	0.7299
0.18	0.3277	0.3657	0.4065	0.4502	0.7285

TABLE VII.17 (Cont'd)

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τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.20	0.3348	0.3720	0.4118	0.4546	0.7268
0.22	0.3417	0.3781	0.4170	0.4589	0.7250
0.24	0.3482	0.3837	0.4218	0.4627	0.7228
0.26	0.3544	0.3891	0.4263	0.4663	0.7204
0.28	0.3603	0.3942	0.4305	0.4696	0.7177
0.30	0.3659	0.3990	0.4345	0.4725	0.7147
0.32	0.3712	0.4035	0.4381	0.4752	0.7114
0.34	0.3762	0.4077	0.4414	0.4776	0.7078
0.36	0.3808	0.4112	0.4443	0.4795	0.7038
0.38	0.3851	0.4149	0.4469	0.4812	0.6994
0.40	0.3889	0.4179	0.4490	0.4823	0.6945
0.42	0.3923	0.4205	0.4506	0.4830	0.6891
0.44	0.3953	0.4226	0.4519	0.4833	0.6832
0.46	0.3977	0.4242	0.4525	0.4829	0.6766
0.48	0.3995	0.4251	0.4525	0.4820	0.6693
0.50	0.4007	0.4254	0.4519	0.4803	0.6612
0.52	0.4010	0.4248	0.4503	0.4777	0.6519
0.54	0.4003	0.4232	0.4477	0.4740	0.6413
0.56	0.3983	0.4202	0.4436	0.4688	0.6289
0.58	0.3942	0.4150	0.4373	0.4613	0.6136
0.60	0.3849	0.4044	0.4253	0.4478	0.5906

TABLE VII.18
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$ $\tau^* = 0.6; \zeta = 45^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.2136	0.2496	0.2882	0.3297	0.5932
0.02	0.2272	0.2622	0.2997	0.3400	0.5962
0.04	0.2381	0.2723	0.3039	0.3483	0.5985
0.06	0.2480	0.2814	0.3172	0.3557	0.6003
0.08	0.2574	0.2901	0.3251	0.3628	0.6021
0.10	0.2662	0.2982	0.3325	0.3693	0.6035
0.12	0.2747	0.3060	0.3336	0.3756	0.6048
0.14	0.2830	0.3137	0.3465	0.3818	0.6061
0.16	0.2909	0.3209	0.3530	0.3876	0.6071
0.18	0.2986	0.3279	0.3594	0.3932	0.6079
0.20	0.3060	0.3347	0.3655	0.3985	0.6085
0.22	0.3132	0.3413	0.3713	0.4036	0.6090
0.24	0.3202	0.3476	0.3770	0.4086	0.6093
0.26	0.3270	0.3538	0.3825	0.4133	0.6094

TABLE VII.18 (Cont'd)

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τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.28	0.3335	0.3597	0.3877	0.4178	0.6093
0.30	0.3398	0.3653	0.3927	0.4221	0.6090
0.32	0.3459	0.3708	0.3975	0.4262	0.6084
0.34	0.3517	0.3760	0.4020	0.4299	0.6076
0.36	0.3572	0.3808	0.4062	0.4334	0.6064
0.38	0.3625	0.3855	0.4102	0.4366	0.6050
0.40	0.3674	0.3898	0.4137	0.4395	0.6032
0.42	0.3719	0.3936	0.4169	0.4419	0.6009
0.44	0.3761	0.3972	0.4198	0.4440	0.5983
0.46	0.3798	0.4002	0.4221	0.4456	0.5950
0.48	0.3829	0.4027	0.4238	0.4465	0.5911
0.50	0.3855	0.4046	0.4250	0.4469	0.5865
0.52	0.3873	0.4057	0.4254	0.4465	0.5809
0.54	0.3883	0.4059	0.4249	0.4452	0.5743
0.56	0.3879	0.4048	0.4229	0.4423	0.5655
0.58	0.3856	0.4017	0.4189	0.4371	0.5549
0.60	0.3781	0.3932	0.4093	0.4266	0.5368

TABLE VII.19

SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$ $\tau^* = 0.6; \zeta = 60^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.1640	0.1868	0.2111	0.2373	0.4039
0.02	0.1756	0.1977	0.2214	0.2469	0.4088
0.04	0.1852	0.2068	0.2299	0.2548	0.4129
0.06	0.1942	0.2153	0.2380	0.2623	0.4168
0.08	0.2028	0.2235	0.2456	0.2694	0.4206
0.10	0.2111	0.2313	0.2530	0.2763	0.4242
0.12	0.2193	0.2391	0.2603	0.2831	0.4279
0.14	0.2273	0.2467	0.2674	0.2897	0.4314
0.16	0.2352	0.2542	0.2745	0.2963	0.4350
0.18	0.2430	0.2615	0.2814	0.3027	0.4384
0.20	0.2508	0.2689	0.2884	0.3092	0.4419
0.22	0.2585	0.2762	0.2952	0.3156	0.4454
0.24	0.2661	0.2834	0.3020	0.3219	0.4488
0.26	0.2736	0.2905	0.3087	0.3282	0.4521
0.28	0.2811	0.2976	0.3153	0.3344	0.4554
0.30	0.2885	0.3046	0.3219	0.3405	0.4586
0.32	0.2958	0.3115	0.3284	0.3465	0.4617
0.34	0.3030	0.3183	0.3348	0.3524	0.4647
0.36	0.3101	0.3250	0.3410	0.3582	0.4676
0.38	0.3171	0.3316	0.3472	0.3639	0.4703
0.40	0.3239	0.3380	0.3532	0.3694	0.4729
0.42	0.3306	0.3443	0.3590	0.3748	0.4753
0.44	0.3371	0.3504	0.3647	0.3800	0.4775
0.46	0.3433	0.3562	0.3700	0.3849	0.4793

TABLE VII.19 (Cont'd)

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τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.48	0.3491	0.3616	0.3750	0.3893	0.4806
0.50	0.3546	0.3666	0.3796	0.3934	0.4816
0.52	0.3595	0.3711	0.3835	0.3969	0.4818
0.54	0.3636	0.3747	0.3867	0.3995	0.4811
0.56	0.3668	0.3775	0.3889	0.4012	0.4792
0.58	0.3682	0.3783	0.3892	0.4009	0.4752
0.60	0.3647	0.3742	0.3844	0.3954	0.4650

TABLE VII.20
SOLUTION OF THE INTEGRAL EQUATION OF THE THEORY OF LIGHT SCATTERING $\varphi_q(\tau)$

$\tau^* = 0.6; \zeta = 76^\circ$

τ	$\varphi_0(\tau)$	$\varphi_{0.1}(\tau)$	$\varphi_{0.2}(\tau)$	$\varphi_{0.3}(\tau)$	$\varphi_{0.8}(\tau)$
0.00	0.0706	0.0790	0.0879	0.0976	0.4039
0.02	0.0767	0.0848	0.0936	0.1029	0.4088
0.04	0.0821	0.0901	0.0986	0.1077	0.4129
0.06	0.0875	0.0953	0.1036	0.1125	0.4168
0.08	0.0928	0.1004	0.1086	0.1173	0.4206
0.10	0.0982	0.1056	0.1136	0.1222	0.4242
0.12	0.1038	0.1111	0.1189	0.1273	0.4279
0.14	0.1095	0.1166	0.1243	0.1325	0.4314
0.16	0.1155	0.1225	0.1299	0.1380	0.4350
0.18	0.1216	0.1284	0.1357	0.1436	0.4384
0.20	0.1280	0.1347	0.1418	0.1495	0.4419
0.22	0.1347	0.1412	0.1482	0.1557	0.4454
0.24	0.1416	0.1480	0.1548	0.1621	0.4488
0.26	0.1488	0.1550	0.1617	0.1689	0.4521
0.28	0.1564	0.1625	0.1690	0.1760	0.4554
0.30	0.1645	0.1704	0.1768	0.1836	0.4586
0.32	0.1728	0.1786	0.1848	0.1915	0.4617
0.34	0.1816	0.1872	0.1933	0.1998	0.4647
0.36	0.1908	0.1963	0.2022	0.2085	0.4676
0.38	0.2004	0.2058	0.2115	0.2176	0.4703
0.40	0.2106	0.2158	0.2214	0.2274	0.4729
0.42	0.2212	0.2263	0.2317	0.2375	0.4753
0.44	0.2324	0.2373	0.2426	0.2482	0.4775
0.46	0.2441	0.2488	0.2539	0.2594	0.4793
0.48	0.2563	0.2609	0.2658	0.2711	0.4806
0.50	0.2690	0.2734	0.2782	0.2833	0.4816
0.52	0.2821	0.2864	0.2909	0.2959	0.4818
0.54	0.2956	0.2997	0.3041	0.3088	0.4811
0.56	0.3091	0.3130	0.3172	0.3217	0.4792
0.58	0.3222	0.3259	0.3299	0.3342	0.4752
0.60	0.3318	0.3353	0.3391	0.3431	0.4650

TABLE VIII
ILLUMINATION OF THE EARTH'S SURFACE BY DIRECT SOLAR RADIATION

τ^*	ζ	$\cos \zeta e^{-\tau^* \sec \zeta}$	τ^*	ζ	$\cos \zeta e^{-\tau^* \sec \zeta}$
0.20	30°	0.68744	0.50	30°	0.48617
	45°	0.53290		45°	0.34865
	60°	0.33516		60°	0.18394
	76°	0.10584		76°	0.03062
0.30	30°	0.61247	0.60	30°	0.43315
	45°	0.46263		45°	0.30267
	60°	0.27441		60°	0.15060
	76°	0.07000		76°	0.02026
0.40	30°	0.54568			
	45°	0.40162			
	60°	0.22466			
	76°	0.04630			

TABLE IX.1
TOTAL ILLUMINATION OF THE EARTH'S SURFACE

$\tau^* = 0.2$

$\zeta \backslash q$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
30°	0.7760	0.7879	0.8002	0.8128	0.8258	0.8393	0.8532	0.8676	0.8824	0.8978	0.9138
45°	0.6191	0.6286	0.6383	0.6483	0.6589	0.6696	0.6807	0.6921	0.7041	0.7163	0.7291
60°	0.4164	0.4228	0.4294	0.4362	0.4432	0.4504	0.4578	0.4653	0.4736	0.4818	0.4904
76°	0.1719	0.1746	0.1773	0.1801	0.1829	0.1859	0.1890	0.1922	0.1955	0.1989	0.2025

TABLE IX.2
TOTAL ILLUMINATION OF THE EARTH'S SURFACE

$\tau^* = 0.3$

$\zeta \backslash q$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
30°	0.7372	0.7528	0.7690	0.7858	0.8036	0.8220	0.8414	0.8618	0.8830	0.9054	0.9290
45°	0.5825	0.5948	0.6076	0.6209	0.6349	0.6495	0.6649	0.6809	0.6977	0.7154	0.7341
60°	0.3841	0.3922	0.4007	0.4095	0.4187	0.4283	0.4384	0.4490	0.4601	0.4717	0.4840
76°	0.1513	0.1545	0.1578	0.1613	0.1649	0.1687	0.1727	0.1769	0.1813	0.1858	0.1907

TABLE IX.3
TOTAL ILLUMINATION OF THE EARTH'S SURFACE

$\tau^* = 0.4$

$\zeta \backslash q$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
30°	0.7019	0.7202	0.7395	0.7599	0.7814	0.8042	0.8283	0.8540	0.8813	0.9103	0.9414
45°	0.5498	0.5642	0.5792	0.5952	0.6121	0.6300	0.6488	0.6689	0.6903	0.7130	0.7374
60°	0.3566	0.3659	0.3757	0.3860	0.3970	0.4086	0.4208	0.4338	0.4477	0.4625	0.4782
76°	0.1360	0.1396	0.1433	0.1472	0.1514	0.1558	0.1605	0.1654	0.1708	0.1764	0.1824

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TABLE IX.4
TOTAL ILLUMINATION OF THE EARTH'S SURFACE

$\zeta \backslash q$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
30°	0.6697	0.6901	0.7118	0.7319	0.7506	0.7860	0.8143	0.8417	0.8773	0.9127	0.9511
45°	0.5200	0.5358	0.5527	0.5706	0.5898	0.6103	0.6322	0.6558	0.6813	0.7088	0.7386
60°	0.3328	0.3480	0.3538	0.3552	0.3774	0.3906	0.4016	0.4198	0.4350	0.4536	0.4726
76°	0.1241	0.1279	0.1319	0.1352	0.1407	0.1457	0.1509	0.1565	0.1626	0.1691	0.1763

TABLE IX.5
TOTAL ILLUMINATION OF THE EARTH'S SURFACE

$\zeta \backslash q$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
30°	0.6399	0.6619	0.6875	0.7109	0.7381	0.7676	0.7995	0.8341	0.8720	0.9134	0.9589
45°	0.4938	0.5103	0.5290	0.5466	0.5696	0.5921	0.6170	0.6437	0.6729	0.7038	0.7400
60°	0.3120	0.3228	0.3312	0.3466	0.3599	0.3742	0.3898	0.4067	0.4252	0.4454	0.4676
76°	0.1148	0.1188	0.1230	0.1276	0.1321	0.1377	0.1431	0.1496	0.1561	0.1638	0.1720

TABLE X.1
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.2; \zeta = 30$

τ	I	Sch	E	T
0.00	0.1984	0.2428	0.2430	0.2611
0.01	0.2003	0.2450	0.2459	0.2689
0.02	0.2031	0.2491	0.2489	0.2748
0.03	0.2051	0.2521	0.2516	0.2801
0.04	0.2073	0.2551	0.2546	0.2848
0.05	0.2102	0.2580	0.2572	0.2892
0.06	0.2127	0.2609	0.2601	0.2933
0.07	0.2152	0.2637	0.2628	0.2970
0.08	0.2176	0.2665	0.2656	0.3005
0.09	0.2202	0.2692	0.2682	0.3037
0.10	0.2227	0.2718	0.2706	0.3067
0.11	0.2253	0.2744	0.2735	0.3094
0.12	0.2279	0.2770	0.2760	0.3119
0.13	0.2306	0.2794	0.2784	0.3140
0.14	0.2333	0.2818	0.2809	0.3158
0.15	0.2360	0.2842	0.2833	0.3174
0.16	0.2387	0.2865	0.2858	0.3185
0.17	0.2415	0.2887	0.2881	0.3191
0.18	0.2443	0.2908	0.2904	0.3191
0.19	0.2471	0.2929	0.2926	0.3184
0.20	0.2500	0.2949	0.2942	0.3153

TABLE X.2
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.2; \zeta = 45$

τ	I	Sch	E	T
0.00	0.1881	0.2316	0.2316	0.2493
0.01	0.1911	0.2352	0.2350	0.2572
0.02	0.1938	0.2386	0.2382	0.2634
0.03	0.1966	0.2420	0.2415	0.2690
0.04	0.1991	0.2451	0.2447	0.2742
0.05	0.2022	0.2488	0.2480	0.2790
0.06	0.2051	0.2521	0.2512	0.2835
0.07	0.2080	0.2554	0.2544	0.2877
0.08	0.2110	0.2586	0.2585	0.2917
0.09	0.2140	0.2618	0.2607	0.2954
0.10	0.2170	0.2649	0.2639	0.2989
0.11	0.2201	0.2680	0.2668	0.3022
0.12	0.2233	0.2711	0.2700	0.3052
0.13	0.2264	0.2741	0.2730	0.3080
0.14	0.2297	0.2771	0.2761	0.3101
0.15	0.2329	0.2800	0.2790	0.3126
0.16	0.2362	0.2829	0.2821	0.3144
0.17	0.2396	0.2857	0.2850	0.3154
0.18	0.2430	0.2885	0.2880	0.3164
0.19	0.2465	0.2912	0.2909	0.3163
0.20	0.2500	0.2939	0.2938	0.3140

TABLE X.3
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.2; \zeta = 60^\circ$

τ	<i>I</i>	<i>Sch</i>	<i>E</i>	<i>T</i>
0.00	0.1676	0.2083	0.2085	0.2246
0.01	0.1710	0.2125	0.2124	0.2328
0.02	0.1744	0.2167	0.2163	0.2396
0.03	0.1779	0.2208	0.2204	0.2458
0.04	0.1815	0.2250	0.2215	0.2517
0.05	0.1852	0.2292	0.2285	0.2572
0.06	0.1889	0.2333	0.2325	0.2626
0.07	0.1928	0.2375	0.2367	0.2678
0.08	0.1967	0.2417	0.2408	0.2727
0.09	0.2006	0.2458	0.2419	0.2775
0.10	0.2047	0.2500	0.2490	0.2821
0.11	0.2088	0.2512	0.2532	0.2866
0.12	0.2130	0.2533	0.2574	0.2908
0.13	0.2173	0.2625	0.2615	0.2948
0.14	0.2217	0.2667	0.2657	0.2986
0.15	0.2262	0.2708	0.2700	0.3021
0.16	0.2308	0.2750	0.2743	0.3054
0.17	0.2354	0.2792	0.2786	0.3081
0.18	0.2402	0.2833	0.2828	0.3104
0.19	0.2450	0.2875	0.2873	0.3119
0.20	0.2500	0.2917	0.2916	0.3112

TABLE X.4
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.2; \zeta = 76^\circ$

τ	<i>I</i>	<i>Sch</i>	<i>E</i>	<i>T</i>
0.00	0.1091	0.1501	0.1427	0.1547
0.01	0.1140	0.1557	0.1479	0.1630
0.02	0.1188	0.1614	0.1531	0.1705
0.03	0.1238	0.1673	0.1586	0.1777
0.04	0.1290	0.1734	0.1611	0.1850
0.05	0.1345	0.1796	0.1699	0.1921
0.06	0.1402	0.1861	0.1758	0.1991
0.07	0.1461	0.1929	0.1820	0.2066
0.08	0.1522	0.1996	0.1861	0.2140
0.09	0.1587	0.2067	0.1950	0.2215
0.10	0.1651	0.2140	0.2018	0.2290
0.11	0.1723	0.2216	0.2089	0.2366
0.12	0.1796	0.2293	0.2162	0.2442
0.13	0.1872	0.2374	0.2237	0.2520
0.14	0.1951	0.2457	0.2315	0.2597
0.15	0.2033	0.2513	0.2394	0.2675
0.16	0.2119	0.2632	0.2476	0.2753
0.17	0.2208	0.2724	0.2567	0.2829
0.18	0.2302	0.2819	0.2656	0.2903
0.19	0.2399	0.2917	0.2749	0.2973
0.20	0.2500	0.3018	0.2846	0.3023

TABLE X.5
APPROXIMATE SOLUTION ACCORDING TO SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.3; \zeta = 60^\circ$

	<i>I</i>	<i>Sch</i>	<i>E</i>	<i>T</i>		<i>I</i>	<i>Sch</i>	<i>E</i>	<i>T</i>
0.00	0.1768	0.2394	0.2396	0.2577	0.16	0.2127	0.2836	0.2834	0.3271
0.01	0.1789	0.2426	0.2426	0.2657	0.17	0.2152	0.2880	0.2858	0.3300
0.02	0.1809	0.2458	0.2455	0.2720	0.18	0.2176	0.2904	0.2882	0.3323
0.03	0.1830	0.2490	0.2484	0.2777	0.19	0.2202	0.2928	0.2906	0.3345
0.04	0.1852	0.2521	0.2512	0.2828	0.20	0.2227	0.2950	0.2929	0.3365
0.05	0.1873	0.2552	0.2542	0.2877	0.21	0.2253	0.2972	0.2954	0.3383
0.06	0.1895	0.2582	0.2569	0.2923	0.22	0.2279	0.2993	0.2975	0.3398
0.07	0.1917	0.2612	0.2598	0.2967	0.23	0.2306	0.3014	0.2997	0.3411
0.08	0.1939	0.2641	0.2625	0.3008	0.24	0.2333	0.3031	0.3019	0.3421
0.09	0.1962	0.2670	0.2652	0.3048	0.25	0.2360	0.3051	0.3040	0.3427
0.10	0.1984	0.2698	0.2679	0.3085	0.26	0.2387	0.3073	0.3061	0.3430
0.11	0.2008	0.2726	0.2705	0.3121	0.27	0.2415	0.3091	0.3083	0.3429
0.12	0.2031	0.2753	0.2731	0.3155	0.28	0.2443	0.3109	0.3102	0.3421
0.13	0.2054	0.2779	0.2758	0.3187	0.29	0.2471	0.3126	0.3128	0.3404
0.14	0.2078	0.2806	0.2783	0.3218	0.30	0.2500	0.3142	0.3142	0.3364
0.15	0.2102	0.2831	0.2803	0.3247					

TABLE X.6
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.3; \zeta = 45^\circ$				
τ	I	Sch	E	T
0.00	0.1636	0.2237	0.2239	0.2410
0.01	0.1659	0.2272	0.2271	0.2490
0.02	0.1683	0.2306	0.2303	0.2551
0.03	0.1706	0.2341	0.2335	0.2612
0.04	0.1731	0.2375	0.2367	0.2666
0.05	0.1756	0.2408	0.2397	0.2717
0.06	0.1780	0.2442	0.2428	0.2765
0.07	0.1806	0.2475	0.2459	0.2812
0.08	0.1832	0.2507	0.2491	0.2857
0.09	0.1858	0.2540	0.2521	0.2900
0.10	0.1884	0.2571	0.2552	0.2941
0.11	0.1911	0.2603	0.2583	0.2980
0.12	0.1938	0.2634	0.2613	0.3019
0.13	0.1966	0.2661	0.2642	0.3056
0.14	0.1994	0.2693	0.2672	0.3091
0.15	0.2022	0.2725	0.2702	0.3126
0.16	0.2051	0.2751	0.2732	0.3158
0.17	0.2080	0.2783	0.2760	0.3188
0.18	0.2110	0.2812	0.2789	0.3218
0.19	0.2140	0.2840	0.2818	0.3245
0.20	0.2170	0.2868	0.2847	0.3271
0.21	0.2201	0.2895	0.2874	0.3291
0.22	0.2233	0.2922	0.2903	0.3316
0.23	0.2261	0.2949	0.2931	0.3335
0.24	0.2297	0.2975	0.2959	0.3352
0.25	0.2339	0.3000	0.2985	0.3366
0.26	0.2362	0.3036	0.3013	0.3376
0.27	0.2396	0.3059	0.3040	0.3381
0.28	0.2430	0.3074	0.3056	0.3381
0.29	0.2465	0.3093	0.3063	0.3372
0.30	0.2500	0.3121	0.3119	0.3369

TABLE X.7
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.3; \zeta = 60^\circ$				
τ	I	Sch	E	T
0.00	0.1372	0.1923	0.1925	0.2075
0.01	0.1400	0.1962	0.1961	0.2153
0.02	0.1428	0.2000	0.1997	0.2218
0.03	0.1457	0.2038	0.2033	0.2278
0.04	0.1486	0.2077	0.2069	0.2331
0.05	0.1516	0.2115	0.2106	0.2399
0.06	0.1547	0.2151	0.2143	0.2443
0.07	0.1578	0.2192	0.2179	0.2491
0.08	0.1610	0.2231	0.2215	0.2545
0.09	0.1643	0.2269	0.2253	0.2594
0.10	0.1676	0.2308	0.2290	0.2642
0.11	0.1710	0.2346	0.2327	0.2689
0.12	0.1744	0.2385	0.2365	0.2734
0.13	0.1779	0.2423	0.2413	0.2780
0.14	0.1815	0.2462	0.2441	0.2825
0.15	0.1852	0.2500	0.2479	0.2868
0.16	0.1889	0.2538	0.2517	0.2911
0.17	0.1928	0.2577	0.2555	0.2952
0.18	0.1967	0.2615	0.2594	0.2992
0.19	0.2006	0.2654	0.2632	0.3032
0.20	0.2047	0.2692	0.2671	0.3069
0.21	0.2088	0.2731	0.2711	0.3107
0.22	0.2130	0.2769	0.2750	0.3150
0.23	0.2173	0.2808	0.2790	0.3173
0.24	0.2217	0.2846	0.2830	0.3204
0.25	0.2262	0.2885	0.2870	0.3235
0.26	0.2308	0.2932	0.2910	0.3268
0.27	0.2351	0.2962	0.2949	0.3279
0.28	0.2402	0.3000	0.2991	0.3299
0.29	0.2450	0.3038	0.3033	0.3309
0.30	0.2500	0.3077	0.3074	0.3288

TABLE X.8
APPROXIMATE SOLUTION ACCORDING TO SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.3; \zeta = 76^\circ$				
τ	I	Sch	E	T
0.00	0.0723	0.1182	0.1136	0.1229
0.01	0.0754	0.1223	0.1173	0.1293
0.02	0.0786	0.1265	0.1211	0.1350
0.03	0.0819	0.1308	0.1250	0.1406
0.04	0.0851	0.1352	0.1290	0.1462
0.05	0.0890	0.1397	0.1331	0.1517
0.06	0.0927	0.1444	0.1373	0.1573
0.07	0.0966	0.1491	0.1416	0.1630
0.08	0.1007	0.1541	0.1462	0.1686
0.09	0.1049	0.1591	0.1508	0.1711
0.10	0.1094	0.1643	0.1557	0.1803
0.11	0.1140	0.1697	0.1606	0.1863
0.12	0.1188	0.1752	0.1657	0.1923
0.13	0.1238	0.1809	0.1710	0.1985
0.14	0.1290	0.1868	0.1765	0.2048
0.15	0.1345	0.1929	0.1821	0.2112

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TABLE X.9
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.4; \zeta = 30^\circ$

τ	I	Sch	E	T
0.00	0.1575	0.2358	0.2362	0.2528
0.02	0.1612	0.2425	0.2422	0.2674
0.04	0.1650	0.2490	0.2479	0.2786
0.06	0.1688	0.2554	0.2537	0.2885
0.08	0.1728	0.2616	0.2593	0.2977
0.10	0.1768	0.2675	0.2649	0.3060
0.12	0.1809	0.2733	0.2702	0.3139
0.14	0.1852	0.2789	0.2755	0.3212
0.16	0.1895	0.2843	0.2806	0.3280
0.18	0.1939	0.2895	0.2857	0.3343
0.20	0.1984	0.2945	0.2906	0.3402
0.22	0.2031	0.2993	0.2952	0.3455
0.24	0.2078	0.3038	0.2999	0.3502
0.26	0.2127	0.3082	0.3043	0.3545
0.28	0.2176	0.3123	0.3087	0.3580
0.30	0.2227	0.3161	0.3129	0.3609
0.32	0.2279	0.3198	0.3170	0.3630
0.34	0.2333	0.3272	0.3203	0.3641
0.36	0.2387	0.3263	0.3246	0.3638
0.38	0.2443	0.3292	0.3281	0.3617
0.40	0.2500	0.3318	0.3314	0.3546

TABLE X.11
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.4; \zeta = 60^\circ$

τ	I	Sch	E	T
0.00	0.1123	0.1786	0.1791	0.1915
0.02	0.1169	0.1857	0.1856	0.2049
0.04	0.1217	0.1929	0.1921	0.2159
0.06	0.1267	0.2000	0.1988	0.2262
0.08	0.1318	0.2071	0.2054	0.2360
0.10	0.1372	0.2143	0.2122	0.2454
0.12	0.1428	0.2214	0.2189	0.2545
0.14	0.1486	0.2286	0.2257	0.2635
0.16	0.1547	0.2357	0.2327	0.2722
0.18	0.1610	0.2429	0.2393	0.2806
0.20	0.1676	0.2500	0.2466	0.2889
0.22	0.1744	0.2571	0.2537	0.2969
0.24	0.1815	0.2643	0.2609	0.3047
0.26	0.1889	0.2714	0.2680	0.3122
0.28	0.1967	0.2786	0.2754	0.3193
0.30	0.2047	0.2857	0.2828	0.3260
0.32	0.2130	0.2929	0.2901	0.3322
0.34	0.2217	0.3000	0.2977	0.3376
0.36	0.2308	0.3071	0.3054	0.3420
0.38	0.2402	0.3143	0.3131	0.3448
0.40	0.2500	0.3214	0.3209	0.3420

TABLE X.10
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.4; \zeta = 45^\circ$

τ	I	Sch	E	T
0.00	0.1420	0.2163	0.2166	0.2319
0.01	0.1461	0.2233	0.2229	0.2462
0.02	0.1503	0.2301	0.2291	0.2574
0.03	0.1543	0.2368	0.2351	0.2676
0.04	0.1590	0.2433	0.2412	0.2770
0.05	0.1636	0.2498	0.2472	0.2858
0.06	0.1682	0.2561	0.2531	0.2942
0.07	0.1731	0.2623	0.2590	0.3021
0.08	0.1780	0.2683	0.2648	0.3096
0.09	0.1832	0.2742	0.2704	0.3167
0.10	0.1884	0.2800	0.2761	0.3234
0.11	0.1938	0.2856	0.2817	0.3297
0.12	0.1994	0.2910	0.2871	0.3355
0.13	0.2051	0.2963	0.2929	0.3408
0.14	0.2110	0.3014	0.2979	0.3456
0.15	0.2170	0.3064	0.3031	0.3497
0.16	0.2232	0.3112	0.3083	0.3531
0.17	0.2297	0.3158	0.3134	0.3556
0.18	0.2362	0.3202	0.3183	0.3568
0.19	0.2430	0.3244	0.3232	0.3562
0.20	0.2500	0.3284	0.3280	0.3508

TABLE X.12
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.4; \zeta = 76^\circ$

τ	I	Sch	E	T
0.00	0.0478	0.0963	0.0987	0.0997
0.02	0.0520	0.1025	0.0992	0.1092
0.04	0.0561	0.1089	0.1049	0.1178
0.06	0.0613	0.1158	0.1110	0.1264
0.08	0.0666	0.1229	0.1175	0.1351
0.10	0.0723	0.1301	0.1243	0.1441
0.12	0.0783	0.1384	0.1315	0.1533
0.14	0.0854	0.1468	0.1392	0.1629
0.16	0.0927	0.1556	0.1494	0.1728
0.18	0.1007	0.1650	0.1559	0.1832
0.20	0.1094	0.1750	0.1652	0.1940
0.22	0.1188	0.1856	0.1750	0.2053
0.24	0.1290	0.1969	0.1856	0.2171
0.26	0.1402	0.2089	0.1967	0.2294
0.28	0.1522	0.2217	0.2088	0.2422
0.30	0.1654	0.2354	0.2217	0.2555
0.32	0.1796	0.2500	0.2356	0.2692
0.34	0.1951	0.2656	0.2503	0.2833
0.36	0.2119	0.2824	0.2663	0.2974
0.38	0.2302	0.3004	0.2834	0.3111
0.40	0.2500	0.3196	0.3018	0.3215

TABLE X.13
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.5; \zeta = 30^\circ$

τ	I	<i>Sch</i>	<i>E</i>	T
0.00	0.1404	0.2323	0.2323	0.2472
0.02	0.1436	0.2391	0.2388	0.2618
0.04	0.1470	0.2458	0.2447	0.2782
0.06	0.1501	0.2523	0.2501	0.2835
0.08	0.1539	0.2587	0.2562	0.2929
0.10	0.1575	0.2649	0.2617	0.3017
0.12	0.1612	0.2710	0.2672	0.3099
0.14	0.1650	0.2768	0.2726	0.3178
0.16	0.1688	0.2825	0.2777	0.3252
0.18	0.1723	0.2880	0.2829	0.3322
0.20	0.1768	0.2934	0.2880	0.3389
0.22	0.1803	0.2985	0.2928	0.3452
0.24	0.1852	0.3034	0.2976	0.3511
0.26	0.1893	0.3082	0.3023	0.3566
0.28	0.1939	0.3128	0.3068	0.3617
0.30	0.1984	0.3171	0.3113	0.3661
0.32	0.2031	0.3212	0.3155	0.3707
0.34	0.2078	0.3251	0.3196	0.3745
0.36	0.2127	0.3288	0.3237	0.3776
0.38	0.2177	0.3323	0.3275	0.3802
0.40	0.2227	0.3355	0.3313	0.3821
0.42	0.2279	0.3385	0.3348	0.3832
0.44	0.2333	0.3412	0.3382	0.3831
0.46	0.2387	0.3437	0.3415	0.3822
0.48	0.2443	0.3460	0.3445	0.3790
0.50	0.2500	0.3480	0.3475	0.3707

TABLE X.14
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.5; \zeta = 45^\circ$

τ	I	<i>Sch</i>	<i>E</i>	T
0.00	0.1233	0.2094	0.2100	0.2228
0.02	0.1268	0.2163	0.2161	0.2367
0.04	0.1304	0.2231	0.2221	0.2478
0.06	0.1342	0.2297	0.2280	0.2580
0.08	0.1380	0.2363	0.2340	0.2674
0.10	0.1420	0.2428	0.2398	0.2763
0.12	0.1461	0.2491	0.2436	0.2849
0.14	0.1503	0.2553	0.2514	0.2931
0.16	0.1546	0.2614	0.2570	0.3009
0.18	0.1590	0.2674	0.2626	0.3084
0.20	0.1636	0.2733	0.2682	0.3157
0.22	0.1682	0.2790	0.2737	0.3226
0.24	0.1731	0.2846	0.2791	0.3293
0.26	0.1780	0.2900	0.2844	0.3356
0.28	0.1832	0.2954	0.2897	0.3416
0.30	0.1884	0.3005	0.2949	0.3473
0.32	0.1933	0.3053	0.3009	0.3536
0.34	0.1984	0.3104	0.3051	0.3574
0.36	0.2031	0.3153	0.3101	0.3618
0.38	0.2110	0.3195	0.3149	0.3657
0.40	0.2170	0.3240	0.3198	0.3696
0.42	0.2232	0.3282	0.3245	0.3715
0.44	0.2297	0.3322	0.3291	0.3731
0.46	0.2362	0.3360	0.3337	0.3735
0.48	0.2430	0.3397	0.3381	0.3720
0.50	0.2500	0.3431	0.3424	0.3654

TABLE X.15
APPROXIMATE SOLUTION ACCORDING TO SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.5; \zeta = 60^\circ$

τ	I	<i>Sch</i>	<i>E</i>	T	τ	I	<i>Sch</i>	<i>E</i>	T
0.0	0.0020	0.1667	0.1674	0.1771	0.26	0.1517	0.2533	0.2481	0.2932
0.02	0.0057	0.1733	0.1734	0.1896	0.28	0.1610	0.2600	0.2550	0.3008
0.04	0.0096	0.1800	0.1791	0.1999	0.30	0.1676	0.2657	0.2617	0.3083
0.06	0.0137	0.1867	0.1854	0.2095	0.32	0.1744	0.2733	0.2681	0.3155
0.08	0.0179	0.1933	0.1915	0.2187	0.34	0.1815	0.2800	0.2752	0.3225
0.10	0.0220	0.2000	0.1977	0.2276	0.36	0.1889	0.2857	0.2821	0.3292
0.12	0.0267	0.2038	0.2363	0.38	0.1957	0.2933	0.2891	0.3357	
0.14	0.0317	0.2133	0.2100	0.2419	0.40	0.2047	0.3000	0.2961	0.3417
0.16	0.0367	0.2200	0.2163	0.2532	0.42	0.2130	0.3067	0.3031	0.3471
0.18	0.0418	0.2267	0.2226	0.2615	0.44	0.2217	0.3133	0.3101	0.3518
0.20	0.0472	0.2333	0.2290	0.2696	0.46	0.2303	0.3200	0.3177	0.3556
0.22	0.0528	0.2400	0.2354	0.2776	0.48	0.2402	0.3267	0.3250	0.3576
0.24	0.0586	0.2467	0.2418	0.2855	0.50	0.2500	0.3333	0.3325	0.3548

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TABLE X.16
APPROXIMATE SOLUTION ACCORDING TO SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.5; \zeta = 76^\circ$

τ	I	<i>Sch</i>	E	T	τ	I	<i>Sch</i>	E	T
0.00	0.0317	0.0810	0.0798	0.0830	0.26	0.0927	0.1641	0.1549	0.1833
0.02	0.0344	0.0857	0.0839	0.0905	0.28	0.1007	0.1735	0.1634	0.1932
0.04	0.0373	0.0907	0.0883	0.0973	0.30	0.1094	0.1833	0.1725	0.2036
0.06	0.0406	0.0959	0.0929	0.1040	0.32	0.1188	0.1936	0.1821	0.2145
0.08	0.0440	0.1013	0.0977	0.1108	0.34	0.1290	0.2047	0.1925	0.2260
0.10	0.0478	0.1070	0.1027	0.1177	0.36	0.1402	0.2161	0.2025	0.2379
0.12	0.0520	0.1129	0.1080	0.1248	0.38	0.1522	0.2290	0.2154	0.2504
0.14	0.0561	0.1191	0.1136	0.1322	0.40	0.1651	0.2425	0.2281	0.2633
0.16	0.0613	0.1257	0.1195	0.1398	0.42	0.1796	0.2568	0.2418	0.2766
0.18	0.0666	0.1326	0.1258	0.1476	0.44	0.1951	0.2723	0.2564	0.2904
0.20	0.0723	0.1399	0.1324	0.1561	0.46	0.2119	0.2888	0.2722	0.3041
0.22	0.0786	0.1476	0.1395	0.1647	0.48	0.2302	0.3065	0.2891	0.3173
0.24	0.0851	0.1557	0.1470	0.1738	0.50	0.2400	0.3256	0.3074	0.3274

TABLE X.17
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.6; \zeta = 30^\circ$

τ	I	<i>Sch</i>	E	T
0.00	0.1250	0.2286	0.2201	0.2411
0.02	0.1280	0.2356	0.2354	0.2556
0.04	0.1310	0.2424	0.2417	0.2670
0.06	0.1340	0.2491	0.2472	0.2773
0.08	0.1371	0.2556	0.2528	0.2868
0.10	0.1404	0.2620	0.2595	0.2978
0.12	0.1436	0.2683	0.2640	0.3014
0.14	0.1470	0.2744	0.2705	0.3125
0.16	0.1504	0.2803	0.2747	0.3202
0.18	0.1539	0.2861	0.2801	0.3277
0.20	0.1575	0.2917	0.2852	0.3348
0.22	0.1612	0.2972	0.2902	0.3417
0.24	0.1650	0.3021	0.2952	0.3482
0.26	0.1683	0.3075	0.2999	0.3511
0.28	0.1728	0.3124	0.3046	0.3603
0.30	0.1768	0.3172	0.3092	0.3659
0.32	0.1809	0.3217	0.3137	0.3712
0.34	0.1852	0.3261	0.3180	0.3762
0.36	0.1895	0.3302	0.3222	0.3803
0.38	0.1939	0.3342	0.3264	0.3850
0.40	0.1984	0.3379	0.3304	0.3889
0.42	0.2031	0.3415	0.3341	0.3923
0.44	0.2078	0.3448	0.3378	0.3952
0.46	0.2127	0.3479	0.3415	0.3977
0.48	0.2176	0.3507	0.3448	0.3995
0.50	0.2227	0.3534	0.3481	0.4006
0.52	0.2279	0.3558	0.3512	0.4010
0.54	0.2333	0.3579	0.3512	0.4003
0.56	0.2387	0.3598	0.3570	0.3983
0.58	0.2443	0.3615	0.3596	0.3942
0.60	0.2500	0.3629	0.3622	0.3849

TABLE X.18
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

$\tau^* = 0.6; \zeta = 45^\circ$

τ	I	<i>Sch</i>	E	T
0.00	0.1070	0.2028	0.2038	0.2130
0.02	0.1101	0.2096	0.2097	0.2272
0.04	0.1132	0.2161	0.2156	0.2331
0.06	0.1165	0.2230	0.2214	0.2430
0.08	0.1198	0.2295	0.2272	0.2574
0.10	0.1233	0.2360	0.2329	0.2662
0.12	0.1268	0.2424	0.2385	0.2747
0.14	0.1301	0.2486	0.2441	0.2846
0.16	0.1342	0.2517	0.2497	0.2996
0.18	0.1380	0.2603	0.2553	0.2986
0.20	0.1420	0.2667	0.2607	0.3061
0.22	0.1461	0.2725	0.2661	0.3132
0.24	0.1503	0.2782	0.2715	0.3202
0.26	0.1546	0.2833	0.2767	0.3270
0.28	0.1590	0.2892	0.2820	0.3325
0.30	0.1636	0.2946	0.2871	0.3398
0.32	0.1682	0.2998	0.2922	0.3459
0.34	0.1731	0.3043	0.2973	0.3517
0.36	0.1780	0.3097	0.3022	0.3572
0.38	0.1832	0.3145	0.3071	0.3625
0.40	0.1881	0.3191	0.3118	0.3674
0.42	0.1938	0.3236	0.3166	0.3719
0.44	0.1994	0.3280	0.3213	0.3761
0.46	0.2051	0.3321	0.3259	0.3798
0.48	0.2110	0.3361	0.3304	0.3829
0.50	0.2170	0.3400	0.3348	0.3855
0.52	0.2232	0.3460	0.3391	0.3873
0.54	0.2297	0.3471	0.3434	0.3883
0.56	0.2362	0.3501	0.3475	0.3879
0.58	0.2430	0.3535	0.3516	0.3856
0.60	0.2500	0.3564	0.3555	0.3781

TABLE X.19
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

τ	I	Sch	E	T
0.00	0.0753	0.1562	0.1381	0.1640
0.02	0.0784	0.1625	0.1435	0.1756
0.04	0.0816	0.1688	0.1490	0.1852
0.06	0.0849	0.1750	0.1556	0.1942
0.08	0.0884	0.1812	0.1615	0.2028
0.10	0.0920	0.1875	0.1674	0.2111
0.12	0.0957	0.1938	0.1734	0.2193
0.14	0.0996	0.2000	0.1794	0.2273
0.16	0.1037	0.2062	0.1854	0.2352
0.18	0.1079	0.2125	0.1915	0.2430
0.20	0.1123	0.2188	0.1977	0.2508
0.22	0.1169	0.2250	0.2038	0.2585
0.24	0.1217	0.2312	0.2100	0.2660
0.26	0.1267	0.2375	0.2163	0.2736
0.28	0.1318	0.2438	0.2226	0.2811
0.30	0.1372	0.2500	0.2290	0.2885
0.32	0.1428	0.2562	0.2354	0.2958
0.34	0.1486	0.2625	0.2418	0.3030
0.36	0.1547	0.2688	0.2484	0.3101
0.38	0.1610	0.2750	0.2550	0.3171
0.40	0.1676	0.2812	0.2617	0.3239
0.42	0.1744	0.2875	0.2684	0.3306
0.44	0.1815	0.2938	0.2752	0.3371
0.46	0.1889	0.3000	0.2821	0.3432
0.48	0.1967	0.3062	0.2891	0.3491
0.50	0.2047	0.3125	0.2961	0.3546
0.52	0.2130	0.3188	0.3031	0.3595
0.54	0.2217	0.3250	0.3104	0.3636
0.56	0.2308	0.3312	0.3177	0.3668
0.58	0.2402	0.3375	0.3250	0.3682
0.60	0.2500	0.3438	0.3325	0.3647

TABLE X.20
APPROXIMATE SOLUTION ACCORDING TO
SCHWARZSCHILD AND EDDINGTON

τ	I	Sch	E	T
0.00	0.0209	0.0700	0.0616	0.0706
0.02	0.0227	0.0738	0.0647	0.0767
0.04	0.0247	0.0777	0.0684	0.0821
0.06	0.0268	0.0818	0.0720	0.0875
0.08	0.0291	0.0860	0.0758	0.0928
0.10	0.0316	0.0904	0.0798	0.0982
0.12	0.0344	0.0950	0.0839	0.1038
0.14	0.0373	0.0998	0.0883	0.1095
0.16	0.0406	0.1048	0.0929	0.1155
0.18	0.0440	0.1100	0.0977	0.1216
0.20	0.0478	0.1155	0.1027	0.1280
0.22	0.0520	0.1212	0.1080	0.1347
0.24	0.0564	0.1273	0.1136	0.1416
0.26	0.0613	0.1337	0.1195	0.1488
0.28	0.0666	0.1404	0.1258	0.1564
0.30	0.0723	0.1475	0.1324	0.1644
0.32	0.0786	0.1550	0.1395	0.1728
0.34	0.0854	0.1629	0.1470	0.1816
0.36	0.0927	0.1714	0.1549	0.1908
0.38	0.1007	0.1804	0.1634	0.2004
0.40	0.1094	0.1899	0.1725	0.2106
0.42	0.1188	0.2001	0.1821	0.2212
0.44	0.1290	0.2109	0.1925	0.2324
0.46	0.1402	0.2225	0.2035	0.2441
0.48	0.1522	0.2349	0.2154	0.2563
0.50	0.1654	0.2482	0.2281	0.2690
0.52	0.1796	0.2623	0.2417	0.2821
0.54	0.1951	0.2776	0.2564	0.2956
0.56	0.2119	0.2959	0.2722	0.3091
0.58	0.2302	0.3114	0.2891	0.3222
0.60	0.2500	0.3303	0.3074	0.3318

TABLE XI.1
APPROXIMATE SOLUTION ACCORDING TO CHANDRASEKHAR
(DIFFUSE RADIATION OF EARTH)

$\tau^* = 0.3$

τ	T	Ch_1	Ch_2	Sch	$\frac{T - Ch_1}{T} \cdot 100$	$\frac{T - Ch_2}{T} \cdot 100$	$\frac{T - Sch}{T} \cdot 100$
0.00	0.6419	0.6031	0.6250	0.6154	6.0%	2.6%	4.3%
0.04	0.5973	0.5756	0.5915	0.5846	3.6%	1.0%	2.2%
0.08	0.5605	0.5481	0.5582	0.5539	2.2%	0.4%	1.2%
0.12	0.5257	0.5206	0.5249	0.5231	1.0%	0.2%	0.5%
0.16	0.4915	0.4931	0.4917	0.4923	-0.3%	-0.04%	-0.2%
0.20	0.4570	0.4656	0.4585	0.4615	-1.9%	-0.3%	-1.0%
0.24	0.4214	0.4381	0.4252	0.4208	-4.0%	-0.9%	-2.2%
0.28	0.3825	0.4106	0.3918	0.4000	-7.3%	-2.4%	-4.4%
0.30	0.3581	0.3969	0.3750	0.3846	-10.8%	-4.7%	-7.4%

TABLE XI.2
APPROXIMATE SOLUTION ACCORDING TO CHANDRASEKHAR
(DIFFUSE RADIATION OF EARTH)

 $\tau^* = 0.5$

τ	T	Ch_1	Ch_2	Sch	$\frac{T - Ch_1}{T} \cdot 100$	$\frac{T - Ch_2}{T} \cdot 100$	$\frac{T - Sch}{T} \cdot 100$
0.00	0.6874	0.6511	0.6756	0.6667	5.3%	1.7%	3.0%
0.02	0.6664	0.6390	0.6613	0.6533	4.1%	0.8%	2.0%
0.06	0.6333	0.6148	0.6329	0.6267	2.9%	0.1%	1.0%
0.10	0.6035	0.5907	0.6047	0.6000	2.1%	-0.2%	0.6%
0.14	0.5751	0.5665	0.5767	0.5733	1.5%	-0.3%	0.3%
0.18	0.5474	0.5423	0.5498	0.5467	0.9%	-0.8%	0.1%
0.22	0.5203	0.5181	0.5209	0.5200	0.4%	-0.1%	0.05%
0.26	0.4933	0.4940	0.4930	0.4933	-0.1%	0.1%	0%
0.30	0.4662	0.4698	0.4652	0.4667	-0.8%	0.2%	-0.1%
0.34	0.4388	0.4456	0.4373	0.4400	-1.5%	0.3%	-0.3%
0.38	0.4109	0.4214	0.4093	0.4133	-2.6%	0.4%	-0.6%
0.42	0.3819	0.3973	0.3812	0.3867	-4.0%	0.2%	-1.3%
0.46	0.3508	0.3731	0.3529	0.3600	-6.4%	-0.6%	-2.6%
0.50	0.3126	0.3489	0.3244	0.3333	-11.6%	-3.8%	-6.6%

TABLE XI.3
APPROXIMATE SOLUTION ACCORDING TO CHANDRASEKHAR
(DIFFUSE RADIATION OF EARTH)

 $\tau^* = 0.6$

τ	T	Ch_1	Ch_2	Sch	$\frac{T - Ch_1}{T} \cdot 100$	$\frac{T - Ch_2}{T} \cdot 100$	$\frac{T - Sch}{T} \cdot 100$
0.00	0.7052	0.6709	0.6954	0.6879	4.9%	1.4%	2.5%
0.04	0.6694	0.6482	0.6589	0.6625	3.2%	0.1%	1.0%
0.08	0.6403	0.6254	0.6425	0.6375	2.3%	-0.3%	0.4%
0.12	0.6133	0.6026	0.6164	0.6125	1.7%	-0.5%	0.1%
0.16	0.5873	0.5798	0.5903	0.5875	1.3%	-0.5%	-0.03%
0.20	0.5620	0.5570	0.5644	0.5625	0.9%	-0.4%	-0.09%
0.24	0.5370	0.5342	0.5386	0.5376	0.5%	-0.3%	-0.1%
0.28	0.5123	0.5114	0.5129	0.5125	0.2%	-0.1%	-0.03%
0.32	0.4877	0.4886	0.4871	0.4875	-0.2%	0.1%	0.04%
0.36	0.4630	0.4658	0.4614	0.4625	-0.6%	0.3%	0.1%
0.40	0.4380	0.4430	0.4356	0.4375	-1.1%	0.5%	0.1%
0.44	0.4127	0.4202	0.4097	0.4125	-1.8%	0.7%	0.04%
0.48	0.3867	0.3974	0.3837	0.3875	-2.8%	0.8%	-0.2%
0.52	0.3597	0.3746	0.3575	0.3625	-4.1%	0.6%	-0.8%
0.56	0.3306	0.3518	0.3311	0.3375	-6.4%	-0.1%	-2.1%
0.60	0.2948	0.3290	0.3046	0.3125	-11.6%	-3.3%	-6.0%

TABLE XI.4
APPROXIMATE SOLUTION ACCORDING TO CHANDRASEKHAR
(DIFFUSE RADIATION OF EARTH)

$\tau^* = 0.3; \zeta = 30^\circ$

τ	T	Sch	Ch_1	Ch_2	$\frac{T-Sch}{T} \cdot 100$	$\frac{T-Ch_1}{T} \cdot 100$	$\frac{T-Ch_2}{T} \cdot 100$
0.00	0.2577	0.2394	0.2308	0.2461	7.1%	10.4%	4.5%
0.02	0.2720	0.2458	0.2369		9.6%	12.9%	
0.04	0.2828	0.2521	0.2426	0.2607	10.8%	14.2%	7.8%
0.06	0.2923	0.2582	0.2184		11.7%	15.0%	
0.08	0.3003	0.2641	0.2539	0.2740	12.2%	15.6%	8.9%
0.10	0.3085	0.2698	0.2593		12.5%	15.9%	
0.12	0.3155	0.2753	0.2647	0.2861	12.7%	16.1%	9.3%
0.14	0.3218	0.2806	0.2699		12.8%	16.1%	
0.16	0.3279	0.2856	0.2750	0.2967	12.9%	16.1%	9.5%
0.18	0.3324	0.2904	0.2798		12.6%	15.8%	
0.20	0.3365	0.2950	0.2847	0.3058	12.3%	15.4%	9.1%
0.22	0.3398	0.2993	0.2889		11.9%	15.0%	
0.24	0.3421	0.3034	0.2933	0.3134	11.3%	14.3%	8.4%
0.26	0.3430	0.3073	0.2976		10.4%	13.2%	
0.28	0.3421	0.3109	0.3017	0.3194	9.1%	11.6%	6.6%
0.30	0.3364	0.3142	0.3056	0.3216	6.6%	9.2%	4.4%

TABLE XI.5
APPROXIMATE SOLUTION ACCORDING TO CHANDRASEKHAR
(DIFFUSE RADIATION OF EARTH)

$\tau^* = 0.3; \zeta = 45^\circ$

τ	T	Sch	Ch_1	Ch_2	$\frac{T-Sch}{T} \cdot 100$	$\frac{T-Ch_1}{T} \cdot 100$	$\frac{T-Ch_2}{T} \cdot 100$
0.00	0.2410	0.2237	0.2160	0.2274	7.1%	10.4%	5.6%
0.02	0.2554	0.2307	0.2223		9.7%	12.9%	
0.04	0.2666	0.2375	0.2276	0.2433	10.9%	14.6%	8.7%
0.06	0.2765	0.2442	0.2349		11.7%	15.0%	
0.08	0.2857	0.2507	0.2411	0.2580	12.2%	15.6%	9.7%
0.10	0.2941	0.2571	0.2472		12.6%	15.9%	
0.12	0.3019	0.2634	0.2532	0.2716	12.7%	16.1%	10.0%
0.14	0.3091	0.2695	0.2592		12.8%	16.1%	
0.16	0.3157	0.2754	0.2650	0.2841	12.8%	16.1%	10.0%
0.18	0.3218	0.2812	0.2708		12.6%	15.8%	
0.20	0.3271	0.2868	0.2765	0.2938	12.3%	15.5%	10.2%
0.22	0.3316	0.2922	0.2822		11.9%	14.9%	
0.24	0.3352	0.2975	0.2877	0.3053	11.2%	14.2%	8.9%
0.26	0.3376	0.3026	0.2931		10.4%	13.2%	
0.28	0.3381	0.3074	0.2984	0.3140	9.1%	11.7%	7.1%
0.30	0.3339	0.3121	0.3037	0.3176	6.5%	9.0%	4.9%

TABLE XI.6
APPROXIMATE SOLUTION ACCORDING TO CHANDRASEKHAR
(DIFFUSE RADIATION OF EARTH)

$\tau^* = 0.3; \zeta = 76^\circ$

τ	T	Sch	Ch_1	Ch_2	$\frac{T - Sch}{T} \cdot 100$	$\frac{T - Ch_1}{T} \cdot 100$	$\frac{T - Ch_2}{T} \cdot 100$
0.00	0.1229	0.1183	0.1080	0.1169	3.7%	12.1%	4.9%
0.02	0.1351	0.1265	0.1155		6.4%	14.5%	
0.04	0.1462	0.1352	0.1233	0.1342	7.5%	15.7%	8.2%
0.06	0.1573	0.1444	0.1317		8.2%	16.3%	
0.08	0.1686	0.1541	0.1405	0.1530	8.6%	16.7%	9.3%
0.10	0.1803	0.1643	0.1500		8.9%	16.8%	
0.12	0.1923	0.1752	0.1600	0.1738	8.9%	16.8%	9.6%
0.14	0.2048	0.1868	0.1708		8.8%	16.6%	
0.16	0.2177	0.1991	0.1822	0.1967	8.5%	16.3%	9.6%
0.18	0.2311	0.2122	0.1945		8.2%	15.8%	
0.20	0.2449	0.2262	0.2076	0.2222	7.6%	15.2%	9.3%
0.22	0.2592	0.2412	0.2216		6.9%	14.5%	
0.24	0.2738	0.2571	0.2366	0.2507	6.1%	13.6%	8.4%
0.26	0.2884	0.2742	0.2528		4.9%	12.3%	
0.28	0.3026	0.2925	0.2701	0.2827	3.3%	10.7%	6.6%
0.30	0.3137	0.3120	0.2888	0.3002	0.5%	7.9%	4.3%

TABLE XI.7
APPROXIMATE SOLUTION ACCORDING TO CHANDRASEKHAR
(DIFFUSE RADIATION OF EARTH)

$\tau^* = 0.5; \zeta = 30^\circ$

τ	T	Sch	Ch_1	Ch_2	$\frac{T - Sch}{T} \cdot 100$	$\frac{T - Ch_1}{T} \cdot 100$	$\frac{T - Ch_2}{T} \cdot 100$
0.00	0.2472	0.2323	0.2202	0.2555	6.0%	10.9%	-3.4%
0.02	0.2619	0.2391	0.2248	0.2633	8.7%	14.2%	-0.5%
0.06	0.2835	0.2523	0.2338	0.2781	11.0%	17.5%	1.9%
0.10	0.3017	0.2649	0.2423	0.2919	12.2%	19.7%	3.2%
0.14	0.3178	0.2768	0.2505	0.3046	12.9%	21.2%	4.2%
0.18	0.3322	0.2880	0.2581	0.3164	13.3%	22.3%	4.8%
0.22	0.3452	0.2985	0.2654	0.3270	13.5%	23.1%	5.3%
0.26	0.3566	0.3082	0.2722	0.3365	13.6%	23.7%	5.6%
0.30	0.3664	0.3171	0.2774	0.3439	13.4%	24.3%	6.1%
0.34	0.3745	0.3251	0.2842	0.3508	13.1%	24.1%	6.8%
0.38	0.3802	0.3323	0.2893	0.3564	12.6%	23.9%	6.3%
0.42	0.3832	0.3385	0.2940	0.3605	11.7%	23.3%	5.9%
0.46	0.3822	0.3437	0.2979	0.3630	10.1%	22.1%	5.0%
0.50	0.3707	0.3480	0.3013	0.3639	6.1%	18.7%	1.8%

TABLE XI.8
APPROXIMATE SOLUTION ACCORDING TO CHANDRASEKHAR
(DIFFUSE RADIATION OF EARTH)

$\tau^* = 0.5; \zeta = 45^\circ$

τ	T	Sch	Ch_1	Ch_2	$\frac{T-Sch}{T} \cdot 100$	$\frac{T-Ch_1}{T} \cdot 100$	$\frac{T-Ch_2}{T} \cdot 100$
0.00	0.2228	0.2094	0.1972	0.2267	6.0%	11.5%	-1.8%
0.02	0.2367	0.2163	0.2019	0.2344	8.6%	14.7%	1.0%
0.06	0.2580	0.2297	0.2093	0.2495	11.0%	18.9%	3.3%
0.10	0.2763	0.2428	0.2165	0.2639	12.1%	21.8%	4.5%
0.14	0.2931	0.2553	0.2234	0.2775	12.9%	23.8%	5.3%
0.18	0.3085	0.2674	0.2301	0.2903	13.3%	25.4%	5.9%
0.22	0.3226	0.2790	0.2366	0.3022	13.5%	26.7%	6.3%
0.26	0.3356	0.2900	0.2427	0.3133	13.6%	27.7%	6.6%
0.30	0.3473	0.3005	0.2486	0.3234	13.5%	28.4%	6.9%
0.34	0.3574	0.3104	0.2541	0.3325	13.2%	28.9%	7.0%
0.38	0.3657	0.3196	0.2594	0.3404	12.6%	29.1%	6.9%
0.42	0.3715	0.3282	0.2644	0.3473	11.7%	28.8%	6.5%
0.46	0.3735	0.3360	0.2689	0.3527	10.0%	28.0%	5.6%
0.50	0.3654	0.3431	0.2731	0.3568	6.1%	25.3%	2.4%

TABLE XII.1
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.2; \zeta = 30^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0070	0.0072	0.0081	0.0099	0.0141	0.0278	0.0123	0.0893
0.04	0.0144	0.0150	0.0167	0.0206	0.0295	0.0592	0.0917	0.2064
0.06	0.0223	0.0231	0.0258	0.0319	0.0459	0.0932	0.1490	0.3586
0.08	0.0301	0.0316	0.0351	0.0438	0.0635	0.1320	0.2151	0.5552
0.10	0.0390	0.0405	0.0454	0.0561	0.0823	0.1748	0.2908	0.8081
0.12	0.0479	0.0497	0.0559	0.0696	0.1022	0.2220	0.3774	1.1320
0.14	0.0572	0.0593	0.0668	0.0833	0.1231	0.2737	0.4760	1.5457
0.16	0.0666	0.0692	0.0780	0.0976	0.1452	0.3301	0.5878	2.0717
0.18	0.0764	0.0794	0.0895	0.1124	0.1683	0.3914	0.7139	2.7366
0.20	0.0863	0.0897	0.1013	0.1276	0.1923	0.4573	0.8549	3.5703

TABLE XII.2
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.2; \zeta = 45^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0084	0.0087	0.0097	0.0119	0.0169	0.0333	0.0505	0.1071
0.04	0.0173	0.0180	0.0201	0.0247	0.0354	0.0710	0.1100	0.2482
0.06	0.0268	0.0278	0.0311	0.0384	0.0553	0.1122	0.1793	0.4322
0.08	0.0368	0.0381	0.0427	0.0529	0.0767	0.1593	0.2595	0.6709
0.10	0.0472	0.0490	0.0550	0.0682	0.0995	0.2115	0.3519	0.9791
0.12	0.0581	0.0603	0.0678	0.0843	0.1239	0.2692	0.4579	1.3759
0.14	0.0694	0.0721	0.0811	0.1012	0.1497	0.3328	0.5791	1.8846
0.16	0.0811	0.0843	0.0950	0.1189	0.1769	0.4024	0.7171	2.5338
0.18	0.0932	0.0969	0.1093	0.1372	0.2056	0.4784	0.8735	3.3583
0.20	0.1056	0.1097	0.1240	0.1561	0.2354	0.5606	1.0491	4.3963

TABLE XII.3
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.2; \zeta = 60^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0113	0.0117	0.0131	0.0160	0.0228	0.0449	0.0682	0.1441
0.04	0.0234	0.0243	0.0228	0.0385	0.0179	0.0961	0.1491	0.3359
0.06	0.0364	0.0378	0.0398	0.0522	0.0752	0.1526	0.2411	0.5883
0.08	0.0502	0.0521	0.0556	0.0723	0.1048	0.2178	0.3550	0.9187
0.10	0.0618	0.0672	0.0727	0.0937	0.1367	0.2907	0.4810	1.3493
0.12	0.0802	0.0832	0.0907	0.1164	0.1710	0.3721	0.6335	1.9087
0.14	0.0963	0.1000	0.1097	0.1405	0.2077	0.4625	0.8060	2.3325
0.16	0.1131	0.1175	0.1324	0.1659	0.2469	0.5626	1.0013	3.5655
0.18	0.1307	0.1358	0.1532	0.1925	0.2885	0.6729	1.2312	4.7625
0.20	0.1437	0.1546	0.1747	0.2201	0.3322	0.7935	1.4887	6.2846

TABLE XII.4
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.2; \zeta = 76^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0191	0.0198	0.0220	0.0272	0.0387	0.0762	0.1176	0.2450
0.04	0.0404	0.0419	0.0467	0.0578	0.0826	0.1659	0.2593	0.5807
0.06	0.0639	0.0663	0.0740	0.0917	0.1320	0.2682	0.4312	1.0372
0.08	0.0897	0.0931	0.1041	0.1292	0.1874	0.3900	0.6386	1.6553
0.10	0.1179	0.1223	0.1372	0.1706	0.2491	0.5311	0.8881	2.4890
0.12	0.1486	0.1543	0.1732	0.2161	0.3177	0.6039	1.1873	3.6084
0.14	0.1820	0.1890	0.2126	0.2660	0.3938	0.8811	1.5447	5.1081
0.16	0.2182	0.2266	0.2553	0.3204	0.4778	1.0960	1.9705	7.1112
0.18	0.2572	0.2673	0.3017	0.3796	0.5703	1.3414	2.4754	9.7751
0.20	0.2990	0.3108	0.3514	0.4435	0.6714	1.6197	3.0704	13.2926

TABLE XII.5
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.3; \zeta = 30^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0073	0.0075	0.0083	0.0104	0.0147	0.0289	0.0443	0.0929
0.04	0.0150	0.0156	0.0173	0.0215	0.0307	0.0616	0.0959	0.2152
0.06	0.0232	0.0241	0.0269	0.0333	0.0480	0.0982	0.1559	0.3746
0.08	0.0319	0.0330	0.0369	0.0459	0.0665	0.1390	0.2254	0.5813
0.10	0.0409	0.0424	0.0476	0.0591	0.0863	0.1842	0.3053	0.8484
0.12	0.0501	0.0523	0.0587	0.0731	0.1074	0.2343	0.3972	1.1925
0.14	0.0602	0.0625	0.0703	0.0878	0.1298	0.2895	0.5026	1.6315
0.16	0.0704	0.0732	0.0824	0.1032	0.1536	0.3503	0.6229	2.2009
0.18	0.0810	0.0842	0.0949	0.1193	0.1787	0.4170	0.7603	2.9449
0.20	0.0920	0.0956	0.1080	0.1361	0.2053	0.4901	0.9164	3.8182
0.22	0.1033	0.1075	0.1215	0.1536	0.2332	0.5699	1.0935	5.0228
0.24	0.1150	0.1196	0.1354	0.1717	0.2625	0.6569	1.2939	6.5126
0.26	0.1269	0.1321	0.1497	0.1904	0.2932	0.7513	1.5199	8.3952
0.28	0.1391	0.1448	0.1645	0.2097	0.3251	0.8532	1.7736	10.7640
0.30	0.1514	0.1577	0.1794	0.2293	0.3580	0.9626	2.0557	13.7190

TABLE XII.6
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.3; \zeta = 45^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0086	0.0089	0.0100	0.0123	0.0174	0.0343	0.0518	0.1103
0.04	0.0179	0.0185	0.0207	0.0255	0.0365	0.0732	0.1132	0.2559
0.06	0.0277	0.0287	0.0321	0.0397	0.0571	0.1169	0.1850	0.4461
0.08	0.0380	0.0394	0.0412	0.0518	0.0793	0.1658	0.2682	0.6944
0.10	0.0489	0.0508	0.0570	0.0708	0.1032	0.2203	0.3646	1.0160
0.12	0.0603	0.0626	0.0704	0.0877	0.1287	0.2808	0.4744	1.4321
0.14	0.0723	0.0751	0.0811	0.1055	0.1559	0.3478	0.6023	1.9694
0.16	0.0848	0.0881	0.0992	0.1243	0.1848	0.4218	0.7490	2.6587
0.18	0.0977	0.1016	0.1146	0.1440	0.2156	0.5034	0.9168	3.5443
0.20	0.1112	0.1156	0.1306	0.1646	0.2482	0.5931	1.1090	4.6781
0.22	0.1252	0.1301	0.1472	0.1861	0.2823	0.6915	1.3273	6.1258
0.24	0.1396	0.1452	0.1645	0.2086	0.3189	0.7991	1.5755	7.9693
0.26	0.1544	0.1607	0.1823	0.2318	0.3570	0.9164	1.8564	10.3096
0.28	0.1696	0.1766	0.2006	0.2559	0.3969	1.0437	2.1729	13.2662
0.30	0.1851	0.1927	0.2192	0.2804	0.4382	1.1808	2.5268	16.9710

TABLE XII.7
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.3; \zeta = 60^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0113	0.0117	0.0131	0.0161	0.0228	0.0450	0.0681	0.1446
0.04	0.0235	0.0244	0.0273	0.0336	0.0480	0.0965	0.1495	0.3373
0.06	0.0366	0.0380	0.0426	0.0525	0.0755	0.1548	0.2451	0.5915
0.08	0.0505	0.0524	0.0588	0.0728	0.1051	0.2206	0.3571	0.9251
0.10	0.0653	0.0678	0.0761	0.0944	0.1378	0.2945	0.4879	1.3616
0.12	0.0809	0.0840	0.0915	0.1176	0.1727	0.3773	0.6379	1.931
0.14	0.0974	0.1022	0.1139	0.1422	0.2102	0.4699	0.8144	2.671
0.16	0.1147	0.1192	0.1344	0.1683	0.2506	0.5729	1.0186	3.632
0.18	0.1330	0.1382	0.1560	0.1959	0.2938	0.6875	1.2544	4.875
0.20	0.1520	0.1581	0.1788	0.2253	0.3399	0.8146	1.5273	6.482
0.22	0.1720	0.1789	0.2026	0.2560	0.3892	0.9553	1.8396	8.552
0.24	0.1928	0.2006	0.2275	0.2884	0.4415	1.1107	2.1976	11.213
0.26	0.2144	0.2232	0.2534	0.3223	0.4970	1.2816	2.6069	14.622
0.28	0.2368	0.2465	0.2803	0.3570	0.5556	1.4689	3.0726	18.989
0.30	0.2597	0.2706	0.3080	0.3943	0.6170	1.6725	3.5983	24.793

TABLE XII.8
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.3; \zeta = 76^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0173	0.0179	0.0201	0.0240	0.0348	0.0687	0.1062	0.2207
0.04	0.0364	0.0377	0.0424	0.0514	0.0743	0.1494	0.2337	0.5226
0.06	0.0575	0.0596	0.0669	0.0819	0.1187	0.2434	0.3879	0.9322
0.08	0.0807	0.0836	0.0932	0.1156	0.1683	0.3525	0.5736	1.4859
0.10	0.1059	0.1098	0.1227	0.1526	0.2235	0.4787	0.7967	2.232
0.12	0.1334	0.1384	0.1550	0.1932	0.2849	0.6243	1.0594	3.234
0.14	0.1633	0.1695	0.1902	0.2378	0.3530	0.7921	1.3791	4.577
0.16	0.1957	0.2032	0.2283	0.2866	0.4284	0.9824	1.7609	6.374
0.18	0.2309	0.2399	0.2705	0.3400	0.5118	1.2010	2.2158	8.776
0.20	0.2690	0.2795	0.3158	0.3983	0.6038	1.4544	2.7575	11.980
0.22	0.3101	0.3224	0.3651	0.4617	0.7053	1.7443	3.4001	16.249
0.24	0.3545	0.3687	0.4182	0.5307	0.8170	2.0756	4.1648	21.927
0.26	0.4022	0.4185	0.4755	0.6061	0.9396	2.4531	5.0689	29.461
0.28	0.4534	0.4720	0.5371	0.6871	1.0737	2.8820	6.1351	39.424
0.30	0.5079	0.5291	0.6030	0.7741	1.2196	3.3660	7.3849	52.516

TABLE XII.9
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.4; \zeta = 30^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0155	0.0161	0.0180	0.0221	0.0317	0.0636	0.0986	0.2223
0.08	0.0330	0.0342	0.0383	0.0474	0.0688	0.1438	0.2328	0.6019
0.12	0.0522	0.0542	0.0609	0.0759	0.1114	0.2431	0.4120	1.2389
0.16	0.0733	0.0761	0.0858	0.1074	0.1598	0.3618	0.6187	2.296
0.20	0.0961	0.0999	0.1128	0.1422	0.2145	0.5125	0.9589	4.038
0.24	0.1206	0.1255	0.1421	0.1802	0.2756	0.6907	1.3624	6.885
0.28	0.1468	0.1528	0.1736	0.2214	0.3436	0.9010	1.8884	11.505
0.32	0.1746	0.1819	0.2072	0.2659	0.4185	1.1574	2.5514	19.348
0.36	0.2037	0.2124	0.2427	0.3133	0.5005	1.4557	3.3991	31.622
0.40	0.2338	0.2440	0.2795	0.3631	0.5885	1.8009	4.4575	50.233

TABLE XII.10
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.4; \zeta = 45^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0182	0.0189	0.0210	0.0260	0.0372	0.0748	0.1160	0.2613
0.08	0.0389	0.0403	0.0451	0.0560	0.0811	0.1698	0.2749	0.7108
0.12	0.0619	0.0642	0.0721	0.0899	0.1320	0.2882	0.4886	1.4704
0.16	0.0872	0.0905	0.1019	0.1278	0.1901	0.4343	0.7728	2.7407
0.20	0.1147	0.1193	0.1347	0.1698	0.2562	0.6130	1.1480	4.847
0.24	0.1446	0.1504	0.1704	0.2161	0.3307	0.8301	1.6396	8.317
0.28	0.1768	0.1840	0.2090	0.2667	0.4141	1.0920	2.2793	13.990
0.32	0.2111	0.2200	0.2506	0.3217	0.5068	1.4056	3.1056	23.697
0.36	0.2474	0.2580	0.2948	0.3809	0.6090	1.7773	4.1630	39.003
0.40	0.2853	0.2977	0.3411	0.4435	0.7197	2.2116	5.4935	62.410

TABLE XIII.11
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.4; \zeta = 60^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0236	0.0213	0.0270	0.0336	0.0478	0.0960	0.1492	0.3356
0.08	0.0503	0.0522	0.0584	0.0726	0.1051	0.2198	0.3564	0.9219
0.12	0.0803	0.0838	0.0941	0.1175	0.1723	0.3766	0.6394	1.928
0.16	0.1148	0.1192	0.1342	0.1683	0.2506	0.5731	1.0218	3.638
0.20	0.1525	0.1585	0.1790	0.2258	0.3409	0.8173	1.535	6.518
0.24	0.1940	0.2018	0.2286	0.2901	0.4414	1.1189	2.218	11.340
0.28	0.2394	0.2492	0.2832	0.3616	0.5622	1.4889	3.122	19.359
0.32	0.2888	0.3009	0.3429	0.4107	0.6955	1.9398	4.310	33.330
0.36	0.3420	0.3566	0.4076	0.5273	0.8452	2.4846	5.858	55.747
0.40	0.3984	0.4158	0.4767	0.6207	1.0104	3.1325	7.845	90.710

TABLE XIII.12
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.4; \zeta = 76^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0283	0.0339	0.0380	0.0464	0.0669	0.1314	0.2096	0.4707
0.08	0.0678	0.0719	0.0810	0.1033	0.1509	0.3160	0.5133	1.332
0.12	0.1147	0.1236	0.1390	0.1725	0.2546	0.5578	0.9496	2.886
0.16	0.1698	0.1810	0.2011	0.2556	0.3816	0.8767	1.570	5.605
0.20	0.2344	0.2483	0.2808	0.3540	0.5363	1.2953	2.449	10.609
0.24	0.3097	0.3269	0.3708	0.4714	0.7241	1.813	3.690	19.378
0.28	0.3974	0.4185	0.4761	0.6095	0.9515	2.557	5.435	31.879
0.32	0.4989	0.5247	0.5989	0.7721	1.2258	3.485	7.882	63.73
0.36	0.6160	0.6475	0.7415	0.9629	1.5555	4.685	11.296	113.17
0.40	0.7497	0.7877	0.9052	1.1838	1.9468	6.221	16.006	196.18

TABLE XIII.13
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.5; \zeta = 30^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0159	0.0165	0.0184	0.0227	0.0325	0.0653	0.1011	0.2282
0.08	0.0339	0.0352	0.0394	0.0488	0.0707	0.1479	0.2393	0.6192
0.12	0.0538	0.0559	0.0628	0.0781	0.1148	0.2505	0.4245	1.278
0.16	0.0757	0.0786	0.0885	0.1109	0.1650	0.3766	0.6700	2.374
0.20	0.0994	0.1033	0.1167	0.1471	0.2219	0.5305	0.9931	4.189
0.24	0.1251	0.1301	0.1474	0.1869	0.2860	0.7172	1.416	7.171
0.28	0.1527	0.1590	0.1806	0.2304	0.3576	0.9422	1.966	12.046
0.32	0.1823	0.1899	0.2164	0.2777	0.4374	1.2120	2.677	19.970
0.36	0.2137	0.2228	0.2546	0.3289	0.5258	1.5338	3.591	32.782
0.40	0.2469	0.2577	0.2953	0.3838	0.6230	1.9150	4.760	53.358
0.44	0.2817	0.2943	0.3382	0.4425	0.7290	2.3634	6.244	85.988
0.48	0.3179	0.3323	0.3830	0.5043	0.8435	2.8851	8.104	137.71
0.50	0.3361	0.3515	0.4057	0.5359	0.9032	3.1724	9.187	173.41

TABLE XII.14
HAZE FACTORS $\sigma(\tau, \theta)$
 $\tau^* = 0.5; \zeta = 45^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0186	0.0192	0.0215	0.0265	0.0379	0.0760	0.1118	0.2657
0.08	0.0396	0.0411	0.0461	0.0571	0.0826	0.1728	0.2798	0.7242
0.12	0.0631	0.0655	0.0737	0.0917	0.1346	0.2939	0.4982	1.501
0.16	0.0891	0.0925	0.1043	0.1305	0.1943	0.4437	0.7898	2.804
0.20	0.1175	0.1221	0.1380	0.1738	0.2623	0.6278	1.1763	4.974
0.24	0.1484	0.1543	0.1749	0.2218	0.3394	0.8524	1.6851	8.564
0.28	0.1819	0.1893	0.2152	0.2745	0.4263	1.1252	2.3513	14.474
0.32	0.2179	0.2270	0.2588	0.3322	0.5237	1.4546	3.2193	24.148
0.36	0.2565	0.2675	0.3058	0.3951	0.6323	1.8501	4.3437	39.871
0.40	0.2977	0.3107	0.3562	0.4632	0.7527	2.3225	5.7921	65.366
0.44	0.3418	0.3563	0.4097	0.5364	0.8851	2.8823	7.6440	106.114
0.48	0.3866	0.4042	0.4661	0.6142	1.0292	3.5388	9.9865	171.221
0.50	0.4093	0.4286	0.4949	0.6543	1.1043	3.9028	11.3605	216.450

TABLE XII.15
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.5; \zeta = 60^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0232	0.0341	0.0271	0.0331	0.0474	0.0952	0.1473	0.3327
0.08	0.0500	0.0518	0.0583	0.0719	0.1042	0.2182	0.3527	0.9147
0.12	0.0802	0.0832	0.0937	0.1164	0.1711	0.3740	0.6338	1.9150
0.16	0.1141	0.1185	0.1337	0.1672	0.2490	0.5696	1.0147	3.6176
0.20	0.1517	0.1577	0.1784	0.2246	0.3392	0.8131	1.5269	6.4938
0.24	0.1923	0.2011	0.2281	0.2890	0.4428	1.1155	2.2111	11.325
0.28	0.2390	0.2488	0.2830	0.3610	0.5615	1.4880	3.1213	19.400
0.32	0.2890	0.3012	0.3435	0.4411	0.6966	1.9450	4.3257	32.829
0.36	0.3485	0.3582	0.4099	0.5298	0.8498	2.5028	5.9117	55.057
0.40	0.4025	0.4202	0.4821	0.6275	1.0226	3.1805	7.9901	91.656
0.44	0.4660	0.4869	0.5603	0.7344	1.2159	3.9981	10.695	151.193
0.48	0.5337	0.5590	0.6442	0.8302	1.4303	4.9752	14.181	248.129
0.50	0.5686	0.5949	0.6877	0.9107	1.5444	5.5248	16.256	316.465

TABLE XII.16
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.5; \zeta = 76^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0297	0.0303	0.0343	0.0424	0.0507	0.1220	0.1893	0.4262
0.08	0.0651	0.0678	0.0738	0.0910	0.1351	0.2353	0.4336	1.2027
0.12	0.1074	0.1113	0.1250	0.1559	0.2292	0.5022	0.8543	2.5922
0.16	0.1565	0.1621	0.1829	0.2291	0.3421	0.7858	1.4065	5.0615
0.20	0.2135	0.2118	0.2507	0.3163	0.4788	1.1557	2.1185	9.4305
0.24	0.2798	0.2909	0.3207	0.4159	0.6439	1.6370	3.2390	17.135
0.28	0.3565	0.3710	0.4219	0.5497	0.8429	2.2521	4.7666	30.701
0.32	0.4451	0.4637	0.5292	0.6816	1.0121	3.0725	6.9030	54.546
0.36	0.5475	0.5711	0.6539	0.8183	1.3705	4.1224	9.8882	96.423
0.40	0.6656	0.6950	0.7985	1.0441	1.7153	5.4798	14.052	169.834
0.44	0.8014	0.8377	0.9659	1.2727	2.1990	7.2301	19.845	297.496
0.48	0.3567	1.0011	1.1584	1.5334	2.6221	9.4783	27.836	520.521
0.50	1.0413	1.0304	1.2641	1.6354	2.8931	10.8043	32.897	686.727

TABLE XII.17
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.6; \zeta = 30^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0009	0.0000	0.0000
0.04	0.0163	0.0169	0.0188	0.0232	0.0332	0.0567	0.1034	0.2331
0.08	0.0347	0.0360	0.0403	0.0499	0.0723	0.1513	0.2448	0.6335
0.12	0.0551	0.0572	0.0612	0.0801	0.1175	0.2566	0.4346	1.3091
0.16	0.0776	0.0805	0.0907	0.1137	0.1692	0.3861	0.6871	2.4382
0.20	0.1021	0.1061	0.1199	0.1511	0.2280	0.5153	1.0205	4.3113
0.24	0.1287	0.1339	0.1517	0.1923	0.2943	0.7386	1.4581	7.4013
0.28	0.1575	0.1639	0.1862	0.2376	0.3689	0.9726	2.0294	12.4721
0.32	0.1893	0.1962	0.2236	0.2870	0.4523	1.2516	2.7720	20.7590
0.36	0.2214	0.2308	0.2638	0.3103	0.5452	1.5930	3.7332	34.2329
0.40	0.2566	0.2678	0.3069	0.3991	0.6482	1.9972	4.9718	56.0534
0.44	0.2939	0.3070	0.3529	0.4619	0.7619	2.4778	6.5603	91.0366
0.48	0.3333	0.3484	0.4017	0.5293	0.8866	3.0463	8.5875	147.438
0.52	0.3746	0.3919	0.4532	0.6011	1.0228	3.7147	11.1584	237.332
0.56	0.4175	0.4372	0.5071	0.6770	1.1701	4.5938	14.3907	379.360
0.60	0.4614	0.4835	0.5626	0.7560	1.3270	5.3875	18.3896	600.075

TABLE XII.18
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.6; \zeta = 45^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0188	0.0194	0.0217	0.0263	0.0383	0.0768	0.1190	0.2686
0.08	0.0401	0.0416	0.0466	0.0578	0.0836	0.1749	0.2829	0.7326
0.12	0.0639	0.0663	0.0745	0.0929	0.1363	0.2976	0.5045	1.5205
0.16	0.0903	0.0937	0.1056	0.1325	0.1970	0.4500	0.8010	2.8156
0.20	0.1192	0.1239	0.1400	0.1766	0.2663	0.6376	1.1950	5.0579
0.24	0.1503	0.1569	0.1778	0.2256	0.3452	0.8672	1.7151	8.7293
0.28	0.1852	0.1928	0.2191	0.2797	0.4343	1.1471	2.3983	14.791
0.32	0.2224	0.2317	0.2610	0.3393	0.5346	1.4865	3.2035	24.765
0.36	0.2621	0.2736	0.3123	0.4044	0.6172	1.8965	4.4589	41.092
0.40	0.3053	0.3187	0.3653	0.4755	0.7729	2.3896	5.9711	67.717
0.44	0.3511	0.3668	0.4218	0.5527	0.9125	2.9800	7.9243	110.705
0.48	0.3999	0.4181	0.4823	0.6361	1.0670	3.6837	10.4341	180.521
0.52	0.4513	0.4724	0.5165	0.7256	1.2367	4.5171	13.6116	202.626
0.56	0.5053	0.5293	0.6142	0.8208	1.4219	5.4965	17.7081	471.200
0.60	0.5609	0.5881	0.6845	0.9209	1.6207	6.6294	22.7813	751.047

TABLE XII.19
HAZE FACTORS $\sigma(\tau, \theta)$

$\tau^* = 0.6; \zeta = 60^\circ$

$\tau \backslash \theta$	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0230	0.0238	0.0266	0.0327	0.0468	0.0940	0.1160	0.3288
0.08	0.0491	0.0512	0.0575	0.0711	0.1030	0.2155	0.3492	0.9044
0.12	0.0793	0.0823	0.0925	0.1152	0.1692	0.3697	0.6274	1.8941
0.16	0.1128	0.1172	0.1321	0.1654	0.2463	0.5633	1.0044	3.5800
0.20	0.1501	0.1560	0.1764	0.2222	0.3357	0.8051	1.5120	6.4314
0.24	0.1915	0.1991	0.2257	0.2862	0.4387	1.1051	2.1918	11.231
0.28	0.2370	0.2467	0.2804	0.3579	0.5568	1.4759	3.0975	19.268
0.32	0.2869	0.2989	0.3408	0.4379	0.6917	1.9321	4.2997	32.676
0.36	0.3415	0.3561	0.4072	0.5268	0.8452	2.4913	5.8894	54.956
0.40	0.4010	0.4185	0.4800	0.6252	1.0193	3.1742	7.9839	91.845
0.44	0.4655	0.4863	0.5596	0.7339	1.2158	4.0052	10.7331	152.37
0.48	0.5352	0.5597	0.6460	0.8532	1.4367	5.0121	14.3263	252.33
0.52	0.6102	0.6387	0.7395	0.9836	1.6839	6.2259	18.9985	415.68
0.56	0.6901	0.7230	0.8399	1.1251	1.9585	7.6780	25.0282	680.60
0.60	0.7743	0.8119	0.9462	1.2764	2.2591	9.3907	32.6984	1103.90

TABLE XII.20
HAZE FACTORS $\sigma(\tau, \theta)$

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$\tau^* = 0.6; \zeta = 76^\circ$

τ	θ	0°	15°	30°	45°	60°	75°	80°	85°
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0273	0.0282	0.0316	0.0389	0.0556	0.1117	0.1710	0.3908	
0.08	0.0597	0.0618	0.0693	0.0859	0.1244	0.2604	0.3731	1.096	
0.12	0.0975	0.1011	0.1136	0.1417	0.2081	0.4556	0.7252	2.350	
0.16	0.1414	0.1468	0.1654	0.2076	0.3092	0.7093	1.2192	4.560	
0.20	0.1921	0.1997	0.2257	0.2849	0.4303	1.0395	1.910	8.449	
0.24	0.2507	0.2603	0.2956	0.3756	0.5768	1.4637	2.924	15.253	
0.28	0.3180	0.3311	0.3764	0.4816	0.7514	2.0121	4.264	27.156	
0.32	0.3954	0.4121	0.4701	0.6056	0.9605	2.7194	6.129	47.981	
0.36	0.4844	0.5053	0.5783	0.7504	1.2108	3.6309	8.722	84.362	
0.40	0.5865	0.6125	0.7036	0.9196	1.5101	4.8052	12.324	147.89	
0.44	0.7039	0.7358	0.8182	1.1171	1.8676	6.3176	17.330	258.25	
0.48	0.8388	0.8777	1.0154	1.3479	2.2949	8.2655	24.284	451.95	
0.52	0.9934	1.0106	1.2083	1.6169	2.8050	10.7705	33.932	789.71	
0.56	1.1702	1.2272	1.4303	1.9297	3.4121	13.9825	47.276	1376.8	
0.60	1.3701	1.4387	1.6831	2.2897	4.1274	19.0693	65.517	2386.7	

The altitude above the earth's surface has been expressed in optical units in the preceding text and Tables covering these investigations. For transition from geometrical to optical altitudes we have used the formula

$$\tau = \int_0^z a(z) dz, \quad (1)$$

where $a(z)$ is the value of the volume scattering coefficient at a height z . The results of the calculations, as given in our Tables, are independent of the particular form of the function $a(z)$ and are applicable for an arbitrary law of distribution of the scattering coefficient with altitude. If, however, it becomes necessary to pass from optical heights to geometric heights then the form of the function $a(z)$ must be known. We present below Tables for this transition, prepared under the assumption that the scattering coefficient varies with altitude by the law of exponents:

$$a(z) = a_0 e^{-kz}, \quad (2)$$

where a_0 is the value of the scattering coefficient at the earth's surface, and k is a constant. Substituting eq.(2) into eq.(1) yields

$$\tau = \frac{a_0}{k} (1 - e^{-kz}). \quad (3)$$

At $z \rightarrow \infty$, this will give the value of the total optical thickness of the atmosphere:

$$\tau^* = \frac{a_0}{k}. \quad (4)$$

Thus,

$$\tau = \tau^* (1 - e^{-kz}), \quad (5)$$

whence

$$z = \frac{1}{k} \log \frac{\tau^*}{\tau^* - \tau}. \quad (6)$$

This is the formula to be used for passing from the optical heights τ to the geometric heights z , if the constants τ^* and k are prescribed. In practice, it is more convenient to replace the constant k by the constant a_0 from eq.(4). In that case, we have

$$z = \frac{\tau^*}{a_0} \log \frac{\tau^*}{\tau^* - \tau}. \quad (7)$$

The quantity a_0 may also be expressed in terms of the horizontal visibility S_h by the following formula, known from the theory of horizontal visibility:

$$S_h := \frac{1}{a_0} \log \frac{1}{\epsilon},$$

where ϵ is the "threshold of contrast sensitivity" ($\epsilon = 0.02$). Hence,

$$a_0 = \frac{1}{S_h} \log \frac{1}{\epsilon}. \quad (9)$$

Thus, we obtain the formula:

$$k = \frac{1}{\tau^* S_h} \log \frac{1}{\epsilon} \quad (10)$$

and

$$z = \tau^* S_h \frac{\log \frac{\tau^*}{\tau}}{\log \frac{1}{\epsilon}}. \quad (11)$$

Since, if $\epsilon = 0.2$, $\log \frac{1}{\epsilon} = 3.912$, eq.(10) for the calculation of k may be written in the form of

$$k = \frac{3.912}{\tau^* S_h}. \quad (12)$$

This reasoning shows that the relation between z and τ is completely determined by the assignment of two constants τ^* and S_h taken from experiment.

The Tables presented below were prepared for five values of τ^* :

$$\begin{array}{c|c|c|c|c|c} \tau^* = 0.2; & | & 0.3; & | & 0.4; & | & 0.5; & | & 0.6 \end{array}$$

and for a number of values of S_h , which latter is always given in kilometers while z is always given in meters. It was unnecessary to take many values of S_h for each τ^* , since a correlation exists between the values of τ^* and S_h : Large values of S_h correspond to small values of τ^* , and small values of S_h to large values of τ^* , although there is no strict functional relation between these quantities.

We note in conclusion that the optical thickness of the atmosphere is related by the following simple formula to the transparency factor p , which is more conventional in practical use:

$$p = e^{-\tau^*}. \quad (13)$$

To the above values of τ^* correspond the following values of the transparency factor:

$$\begin{array}{c|c|c|c|c|c} \tau^* & | & 0.2 & | & 0.3 & | & 0.4 & | & 0.5 & | & 0.6 \\ p & | & 0.819 & | & 0.741 & | & 0.670 & | & 0.607 & | & 0.549 \end{array}$$

VALUES OF k

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s_h	τ^*	0.20	0.30	0.40	0.50	0.60
10		1.95600	1.30400	0.97800	0.78210	0.65200
20		0.97800	0.65200	0.48900	0.39120	0.32600
30		0.65200	0.43167	0.32600	0.26747	0.21733
40		0.48900	0.32600	0.24450	0.19560	0.16300
50		0.39120	0.26747	0.19560	0.15648	0.13040
60		0.32600	0.21733	0.16300	0.13040	0.10867
70		0.27913	0.18629	0.13971	0.11177	0.09314
80		0.24450	0.16300	0.12225	0.09780	0.08150
90		0.21733	0.14189	0.10867	0.08693	0.07241
100		0.19560	0.13040	0.09780	0.07824	0.06520

TRANSFER TABLES

$\tau^* = 0.20$

τ	$s_h = 30$	$s_h = 40$	$s_h = 50$	$s_h = 60$	$s_h = 70$	$s_h = 80$	$s_h = 90$
0	0	0	0	0	0	0	0
0.01	80	107	134	160	187	214	240
0.02	162	216	269	323	377	431	485
0.03	249	332	415	499	582	665	748
0.04	342	456	570	684	799	913	1027
0.05	441	588	735	882	1029	1177	1324
0.06	547	729	912	1094	1276	1459	1641
0.07	661	881	1101	1321	1542	1762	1982
0.08	784	1045	1306	1567	1828	2089	2350
0.09	917	1223	1528	1833	2139	2445	2751
0.10	1063	1418	1772	2126	2481	2835	3189
0.11	1225	1633	2041	2449	2853	3266	3674
0.12	1405	1863	2342	2811	3279	3748	4216
0.13	1610	2136	2684	3220	3757	4294	4830
0.14	1847	2451	3078	3693	4309	4924	5540
0.15	2126	2824	3544	4252	4961	5670	6379
0.16	2468	3280	4114	4937	5760	6583	7405
0.17	2910	3869	4850	5819	6789	7759	8729
0.18	3532	4709	5896	7063	8240	9418	10595
0.190	4595	6126	7658	9189	10721	12252	13784
0.191	4756	6342	7927	9513	11098	12683	14269
0.192	4937	6583	8228	9874	11519	13165	14811
0.193	5142	6856	8570	10281	11997	13711	15425
0.194	5378	7171	8964	10756	12549	14342	16135
0.195	5658	7544	9430	11316	13201	15087	16974
0.196	6000	8000	10000	12000	14000	16000	18000
0.197	6441	8588	10736	12883	15030	17177	19324
0.198	7063	9418	11772	14126	16481	18835	21190
0.199	8126	10335	13544	16252	18961	21670	24379

$\tau^* = 0.30$

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τ	$S_h = 30$	$S_h = 40$	$S_h = 50$	$S_h = 60$	τ	$S_h = 30$	$S_h = 40$	$S_h = 50$	$S_h = 60$
0	0	0	0	0	0.20	2527	3370	4107	5035
0.01	78	104	127	156	0.21	2770	3693	4501	5516
0.02	159	212	258	317	0.22	3041	4054	4942	6081
0.03	242	323	394	485	0.23	3348	4464	5441	6696
0.04	329	440	535	658	0.24	3703	4937	6017	7406
0.05	419	559	682	839	0.25	4122	5496	6699	8241
0.06	513	684	834	1027	0.26	4635	6181	7533	9271
0.07	611	815	993	1223	0.27	5297	7063	8609	10395
0.08	714	951	1160	1427	0.28	6230	8307	10125	12461
0.09	821	1094	1333	1535	0.29	7825	10433	12716	15650
0.10	933	1244	1516	1866	0.291	8067	10756	13110	16135
0.11	1051	1401	1748	2102	0.292	8338	11118	13550	16677
0.12	1175	1567	1910	2350	0.293	8645	11527	14626	17291
0.13	1307	1742	2123	2618	0.294	9000	12000	15000	18000
0.14	1446	1928	2350	2892	0.295	9419	12559	15308	18339
0.15	1595	2126	2591	3189	0.296	9933	13244	16142	19366
0.16	1753	2338	2849	3507	0.297	10595	14126	17218	
0.17	1924	2565	3126	3848	0.298	11527	15370	18733	
0.18	2108	2811	3426	4216	0.299	13122	17496	21325	
0.19	2308	3078	3751	4617					

 $\tau^* = 0.40$

τ	$S_h = 20$	$S_h = 30$	$S_h = 40$	τ	$S_h = 20$	$S_h = 30$	$S_h = 40$
0	0	0	0	0.25	2007	3010	4013
0.01	52	77	103	0.26	2147	3220	4294
0.02	105	157	210	0.27	2298	3448	4597
0.03	159	239	319	0.28	2462	3693	4924
0.04	215	323	431	0.29	2640	3960	5280
0.05	273	410	546	0.30	2835	4252	5670
0.06	332	499	665	0.31	3050	4576	6101
0.07	393	590	787	0.32	3291	4937	6583
0.08	456	684	913	0.33	3564	5346	7129
0.09	521	782	1042	0.34	3880	5819	7759
0.10	588	882	1176	0.35	4232	6379	8505
0.11	653	986	1315	0.36	4709	7063	9417
0.12	729	1094	1459	0.37	5297	7945	10594
0.13	804	1206	1608	0.38	6126	9189	12252
0.14	881	1321	1762	0.390	7544	11316	15087
0.15	961	1442	1922	0.391	7759	11639	15518
0.16	1045	1567	2089	0.392	8000	12000	16000
0.17	1132	1697	2263	0.393	8273	12410	16546
0.18	1223	1834	2445	0.394	8588	12892	17177
0.19	1318	1977	2635	0.395	8961	13442	17922
0.20	1417	2126	2835	0.396	9417	14126	18835
0.21	1522	2284	3045	0.397	10006	15009	20012
0.22	1633	2450	3266	0.398	10835	16252	21670
0.23	1750	2625	3500	0.399	12252	18379	24505
0.24	1874	2811	3748				

$\tau^* = 0.50$

τ	$S_h = 20$	$S_h = 30$	τ	$S_h = 20$	$S_h = 30$
0	0	0			
0.01	52	74	0.30	2342	3424
0.02	104	151	0.31	2473	3015
0.03	158	230	0.32	2612	3818
0.04	213	311	0.33	2753	4031
0.05	275	401	0.34	2913	4253
0.06	327	477	0.35	3078	4499
0.07	385	563	0.36	3254	4757
0.08	446	651	0.37	3443	5034
0.09	507	741	0.38	3648	5333
0.10	570	833	0.39	3870	5659
0.11	694	1014	0.40	4114	6015
0.12	701	1025	0.41	4383	6109
0.13	770	1125	0.42	4684	6819
0.14	840	1227	0.43	5026	7349
0.15	912	1332	0.44	5420	7925
0.16	986	1441	0.45	5886	8609
0.17	1062	1552	0.46	6456	9443
0.18	1147	1676	0.47	7192	10519
0.19	1222	1786	0.48	8228	12035
0.20	1306	1909	0.490	10000	13626
0.21	1392	2035	0.491	10269	14020
0.22	1482	2167	0.492	10570	14460
0.23	1575	2303	0.493	10912	14977
0.24	1672	2444	0.494	11306	15536
0.25	1772	2590	0.495	11772	16218
0.26	1876	2743	0.496	12342	17052
0.27	1985	2902	0.497	13078	18127
0.28	2099	3068	0.498	14114	19643
0.29	2217	3241	0.499	15886	23235

 $\tau^* = 0.60$

τ	$S_h = 10$	$S_h = 20$	$S_h = 30$	τ	$S_h = 10$	$S_h = 20$	$S_h = 30$
0	0	0	0	0.14	407	815	1222
0.01	25	51	77	0.15	441	882	1324
0.02	52	104	156	0.16	476	951	1427
0.03	79	157	236	0.17	511	1022	1533
0.04	106	212	317	0.18	547	1094	1641
0.05	133	267	400	0.19	584	1168	1752
0.06	161	323	485	0.20	625	1249	1874
0.07	190	381	571	0.21	661	1322	1982
0.08	220	439	659	0.22	701	1401	2102
0.09	249	499	748	0.23	741	1483	2224
0.10	280	559	839	0.24	783	1567	2351
0.11	311	621	932	0.25	827	1653	2480
0.12	342	684	1026	0.26	871	1742	2613
0.13	375	749	1124	0.27	917	1834	2751

$\tau^* = 0.60$

(continued)

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τ	$S_h = 10$	$S_h = 20$	$S_h = 30$	τ	$S_h = 10$	$S_h = 20$	$S_h = 30$
0.28	964	1928	2892	0.49	2603	5204	7806
0.29	1013	2026	3039	0.50	2749	5496	8244
0.30	1063	2126	3189	0.51	2911	5819	8729
0.31	1115	2230	3345	0.52	3091	6181	9271
0.32	1169	2338	3507	0.53	3296	6590	9886
0.33	1225	2449	3674	0.54	3532	7063	10395
0.34	1283	2555	3848	0.55	3811	7622	11434
0.35	1343	2655	4028	0.56	4153	8307	12461
0.36	1405	2811	4216	0.57	4595	9189	13784
0.37	1471	2941	4412	0.58	5217	10433	15650
0.38	1539	3078	4616	0.590	6280	12559	18839
0.39	1610	3220	4830	0.591	6441	12883	19324
0.40	1685	3340	5055	0.592	6622	13244	19866
0.41	1764	3527	5291	0.593	6827	13553	20481
0.42	1847	3693	5540	0.594	7063	14126	21190
0.43	1934	3838	5803	0.595	7343	14686	22029
0.44	2027	4051	6082	0.596	7685	15370	23055
0.45	2127	4252	6379	0.597	8126	16252	24379
0.46	2233	4464	6696	0.598	8748	17496	26245
0.47	2347	4691	7037	0.599	9811	19623	29434
0.48	2469	4937	7406				

II. GRAPH FOR COMPARING THE EXACT SOLUTION WITH
CHANDRASEKHAR'S RESULTS

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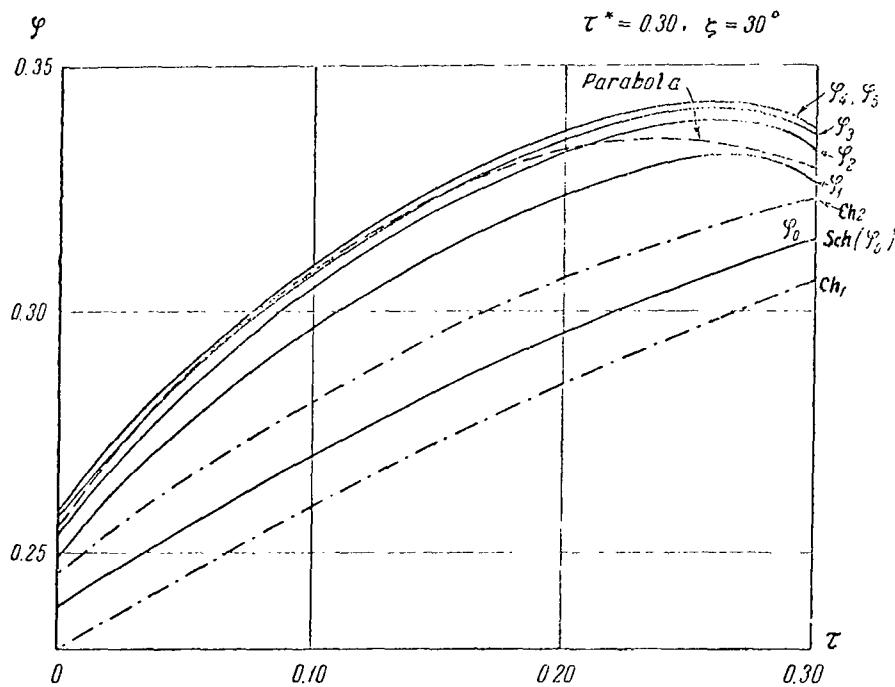


Fig.1 Comparison of the Exact Solution with
Chandrasekhar's Results

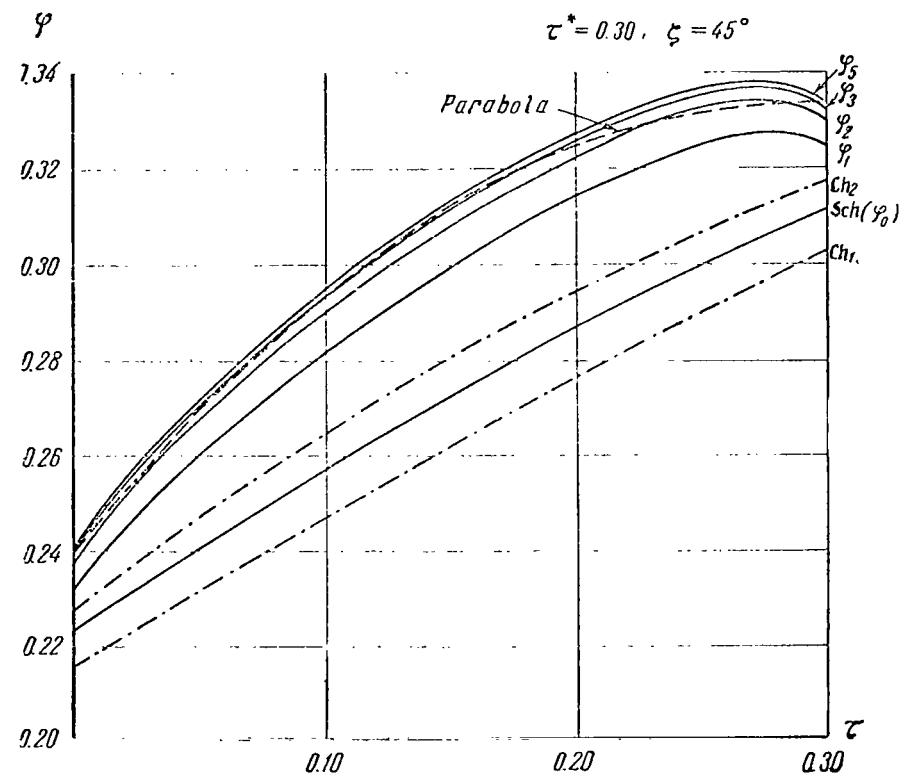


Fig.2 Comparison of the Exact Solution with Chandrasekhar's Results

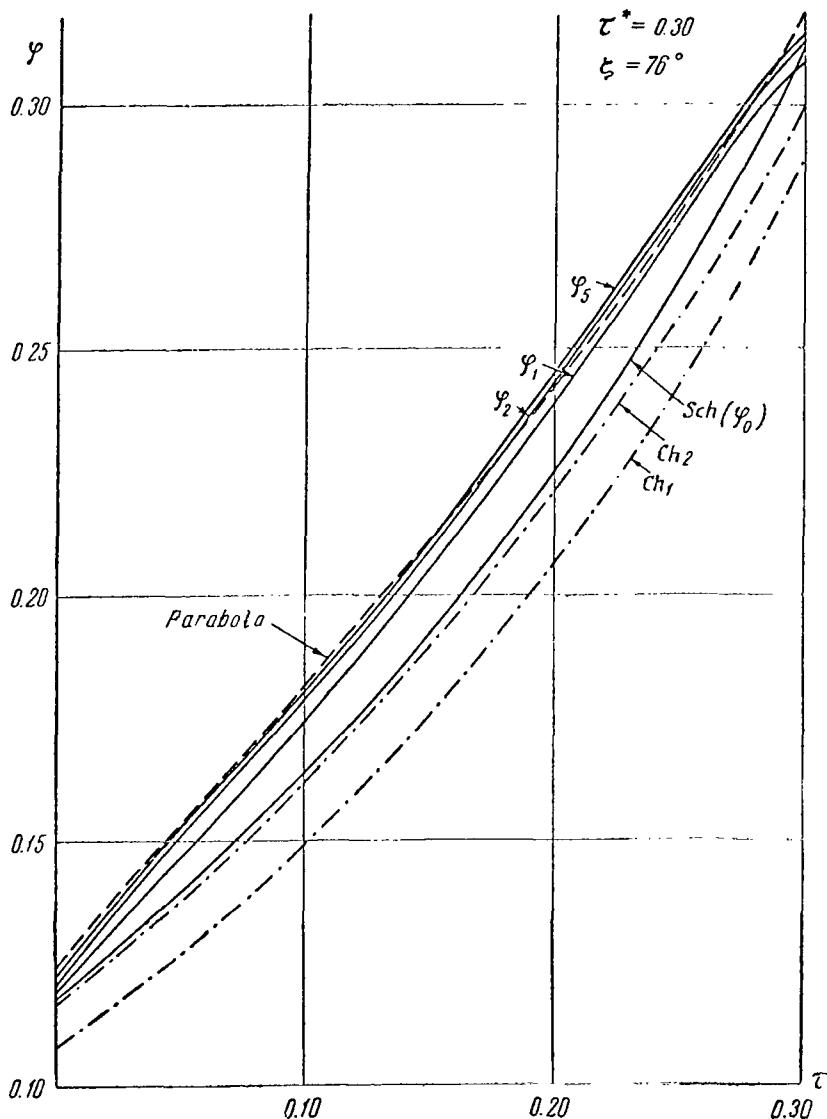


Fig.3 Comparison of the Exact Solution with Chandrasekhar's Results

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